

Two-point correlation function of three-dimensional $O(N)$ models: The critical limit and anisotropy

Massimo Campostrini, Andrea Pelissetto, Paolo Rossi, and Ettore Vicari
Dipartimento di Fisica dell'Università and INFN, I-56126 Pisa, Italy

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In three-dimensional $O(N)$ models, we investigate the low-momentum behavior of the two-point Green's function $G(x)$ in the critical region of the symmetric phase. We consider physical systems whose criticality is characterized by a rotationally invariant fixed point. Several approaches are exploited, such as strong-coupling expansion of lattice N -vector model, and $1/N$ expansion, field-theoretical methods within the ϕ^4 continuum formulation. Non-Gaussian corrections to the universal low-momentum behavior of $G(x)$ are evaluated, and found to be very small. In nonrotationally invariant physical systems with $O(N)$ -invariant interactions, the vanishing of the spatial anisotropy approaching the rotationally invariant fixed point is described by a critical exponent ρ , which is universal and is related to the leading irrelevant operator breaking rotational invariance. At $N=\infty$ one finds $\rho=2$. We show that, for all values of $N\geq 0$, $\rho=2$. [S1063-651X(98)03301-7]

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I. INTRODUCTION

Three-dimensional $O(N)$ models describe many important critical phenomena in nature. The statistical properties of ferromagnetic materials are described by the case $N=3$, where the Lagrangian field represents the magnetization. The helium superfluid transition, whose order parameter is the complex quantum amplitude of helium atoms, corresponds to $N=2$. The case $N=1$ (i.e., Ising-like systems) describes liquid-vapor transitions in classical fluids or critical binary fluids, where the order parameter is the density. $O(N)$ models in the limit $N\rightarrow 0$ describe the statistical properties of long polymers.

The critical behavior of the two-point correlation function $G(x)$ of the order parameter is relevant in the description of critical scattering observed in many experiments, typically neutron scattering in ferromagnetic materials, light and x rays in liquid-gas systems, etc. In Born's approximation the cross section Γ_{fi} for incoming particles (i.e., neutrons or photons) of momentum p_i and final outgoing momentum p_f is proportional to the component $k=p_f-p_i$ of the Fourier transform of $G(x)$:

$$\Gamma_{fi} \propto \tilde{G}(k=p_f-p_i). \quad (1)$$

As a consequence of the critical behavior of the two-point function $G(x)$ at T_c ,

$$\tilde{G}(k) \sim \frac{1}{k^{2-\eta}}, \quad (2)$$

the cross section for $k\rightarrow 0$ (forward scattering) diverges as $T\rightarrow T_c$. When strictly at criticality the relation (2) holds at all momentum scales. In the vicinity of the critical point where the relevant correlation length ξ is large but finite, the behavior (2) occurs for $\Lambda \gg k \gg 1/\xi$, where Λ is a generic cutoff related to the microscopic structure of the statistical system, for example, the inverse lattice spacing in the case of

lattice models. At low momentum, $k \ll 1/\xi$, experiments show that $G(x)$ is well approximated by a Gaussian behavior,

$$\frac{\tilde{G}(0)}{\tilde{G}(k)} \simeq 1 + \frac{k^2}{M_G^2}, \quad (3)$$

where $M_G \sim 1/\xi$ is a mass scale defined at zero momentum (for a general discussion see, e.g., Ref. [1]).

In this paper we will consider three-dimensional systems with an $O(N)$ -invariant Hamiltonian in the symmetric phase, where the $O(N)$ symmetry is unbroken. We will study the two-point correlation function of the order parameter, the Lagrangian field, focusing mainly on its low-momentum behavior. We will estimate the deviations from Eq. (3). We will focus on two quite different sources of deviations:

(i) Scaling corrections to Eq. (3), depending on the ratio k^2/M_G^2 , and reflecting the non-Gaussian nature of the fixed point.

(ii) Nonrotationally invariant scaling violations, reflecting a microscopic anisotropy in the space distribution of the spins. This phenomenon may be relevant, for example, in the study of ferromagnetic materials, where the atoms lie on the sites of a lattice giving rise to a spatial anisotropy which may be observed in neutron-scattering experiments. In these systems the anisotropy vanishes in the critical limit, and $G(x)$ approaches a rotationally invariant form. It should be noticed that this phenomenon is different from the breakdown of the $O(N)$ symmetry in the interaction, which has been widely considered in the literature [2].

In our study of the critical behavior of the two-point function of the order parameter $G(x)$ we will consider several approaches. We analyze the strong-coupling expansion of $G(x)$,

$$G(x) \equiv \langle \vec{s}(x) \cdot \vec{s}(0) \rangle, \quad (4)$$

for the lattice N -vector [$O(N)$ nonlinear σ] model with nearest-neighbor interactions

$$S_L = -N\beta \sum_{\text{links } \langle xy \rangle} \vec{s}_x \cdot \vec{s}_y, \quad (5)$$

which we have calculated up to 15th order on the simple cubic lattice and 21st order on the diamond lattice. We also perform a detailed study using the $1/N$ expansion, whose results, beside clarifying physical mechanisms, are also useful as benchmarks for the strong-coupling analysis. Moreover we compute the first few nontrivial terms of the ϵ expansion and of the g expansion (i.e., expansion in the four-point renormalized coupling at fixed dimension $d=3$) of the two-point function for the corresponding ϕ^4 continuum formulation of $O(N)$ models:

$$\mathcal{L}_{\phi^4} = \frac{1}{2} \partial_\mu \phi(x) \partial_\mu \phi(x) + \frac{1}{2} \mu_0^2 \phi^2 + \frac{1}{4!} g_0 (\phi^2)^2. \quad (6)$$

We recall that the N -vector model and the ϕ^4 model with the same internal symmetry $O(N)$ describe the same critical behavior. By universality our study provides information on the behavior of the physical systems mentioned above in the critical region of the high-temperature phase. A short report of our study can be found in Ref. [3].

The first systematic study of the critical behavior of $G(x)$ is due to Fisher and Aharony [1,4,5]. They computed $G(x)$ in the ϵ expansion up to terms $O(\epsilon^2)$ [4] and in the large- N expansion to order $1/N$ [5]; moreover some estimates of the non-Gaussian corrections for $N=1$ and $N=3$ were derived from strong-coupling series for various lattices [4,6]. Their calculations confirmed experimental observations that non-Gaussian corrections are small in the low-momentum region.

In this paper we reconsider the problem of calculating the two-point function $G(x)$ in the low-momentum regime using the different approaches we mentioned above. We show that the low-momentum expansion around $k^2=0$ of the scaling two-point function provides a very good approximation in a relatively large range of momenta, up to $|k| \lesssim 3M_G$.

We compute the expansion of the scaling two-point function and of its low-momentum expansion up to four loops, $O(g^4)$, in fixed dimension $d=3$ and we extend the results of Ref. [4] by calculating the next three-loop term $O(\epsilon^3)$. We improve earlier strong-coupling calculations concerning the low-momentum expansion of $G(x)$. This is achieved essentially for two reasons: longer strong-coupling series are available, and, more importantly, we consider improved estimators that allow more stable extrapolations to the critical limit. The results of the various approaches are reasonably consistent among each other: the g expansion and the analysis of the strong-coupling series provide in general the most precise estimates, together with the $1/N$ expansion for $N \gtrsim 16$. The ϵ expansion is somewhat worse but still consistent, perhaps because of the limited number of terms (one term less than in the g expansion).

We also discuss the spatial anisotropy in $G(x)$ induced by the lattice structure. For the class of systems we consider, $G(x)$ becomes rotationally invariant at criticality: when $\beta \rightarrow \beta_c$, so that $M_G \rightarrow 0$, the anisotropic deviations vanish as M_G^ρ , where ρ is a universal critical exponent. From a field-theoretical point of view, the spatial anisotropy is due to nonrotationally invariant $O(N)$ -symmetric irrelevant operators in the effective Hamiltonian, whose presence depends

essentially on the symmetries of the physical system or of the lattice formulation. The exponent ρ is related to the critical effective dimension of the leading irrelevant operator breaking rotational invariance. On d -dimensional lattices with cubic symmetry the leading operator has canonical dimension $d+2$. In the large- N limit, where the canonical dimensions determine the scaling properties, one finds $\rho=2$ with very small $O(1/N)$ corrections. A strong-coupling analysis supported by a two-loop ϵ expansion and three-loop g expansion computation indicate that ρ remains close to its canonical value for all $N \geq 0$, with deviations of approximately 1% for small values of N . It should be noted that the exponent ρ , which controls the recovery of rotational invariance, is different from ω , the leading subleading exponent, since they are related to different irrelevant operators. This means—and this may be of relevance for numerical calculations—that the recovery of rotational invariance is unrelated to the disappearance of the subleading corrections controlled by ω : in practice, as $\rho \approx 2$ while $0.8 \lesssim \omega \lesssim 1$ [2,7] ($\omega \approx 0.80$ for $N=0,1,2,3$), rotational invariance is recovered long before the scaling region.

We also investigated the recovery of rotational invariance in two-dimensional models. On the square lattice, for $N=1$ (Ising model) and $N \geq 3$, we show that $\rho=2$. This leads us to conjecture that $\rho=2$ holds exactly for all two-dimensional models on the square lattice. Similarly we conjecture that $\rho=4$ ($\rho=3$) are the exact values of the exponents for the triangular (honeycomb) lattice. A Monte Carlo and exact-enumeration study [8] for $N=0$ on the square lattice is consistent with this conjecture. We should mention that our results on the spatial anisotropy are also relevant in the discussion of the linear response of the system in presence of an external (anisotropic) field.

The paper is organized as follows: In Sec. II we fix the notation and introduce a general parametrization of $G(x)$ that includes the off-critical and nonspherical dependence. In Sec. III we analyze the critical behavior of $G(x)$ at low momentum. We present calculations based on various approaches: $1/N$ expansion [up to $O(1/N)$], g expansion [up to $O(g^4)$], ϵ expansion [up to $O(\epsilon^3)$], and an analysis of the strong-coupling expansion of $G(x)$ on the cubic and diamond lattice. In Sec. IV the anisotropy of $G(x)$ is studied in the critical region. We present large- N and $O(1/N)$ calculations on various lattices, and a strong-coupling analysis of some nonspherical moments of $G(x)$ on cubic and diamond lattices. Again, the analysis of the first nontrivial terms of the g expansion and the ϵ expansion is presented. Anisotropy in $G(x)$ is also studied in two-dimensional $O(N)$ models. In Appendix A we present some details of our $O(1/N)$ calculations. In Appendix B we present the 15th-order strong-coupling expansion of the two-point function on the cubic lattice for selected values of N . In Appendix C we report the 21st-order strong-coupling series of the magnetic susceptibility and of the second moment of $G(x)$ on the diamond lattice for $N=1,2,3$.

II. THE TWO-POINT GREEN'S FUNCTION

A. Hypercubic lattices

In this section we discuss the general behavior of the two-point spin-spin correlation function in lattice $O(N)$ models. We consider a generic Hamiltonian defined on a hypercubic lattice,

$$\mathcal{H} = - \sum_{x,y} J(x-y) \vec{s}_x \cdot \vec{s}_y, \quad (7)$$

where the sum runs over all lattice sites. Below we will extend our analysis to other lattices. Let us define

$$\bar{k}^2(k) = 2[\tilde{J}(k) - \tilde{J}(0)], \quad (8)$$

where $\tilde{J}(k)$ denotes the Fourier transform of $J(x)$. In spite of the notation, we are not assuming that $\bar{k}^2(k)$ is a sum of the type $\sum_{\mu} f(k_{\mu})$. We consider models for which, by a suitable normalization of the inverse temperature β ,

$$\bar{k}^2(k) = k^2 + O(k^4), \quad (9)$$

so that the critical limit is rotationally invariant. Moreover we make the following assumptions: (1) The interaction $J(x)$ is short ranged so that \bar{k}^2 is continuous; (2) the function $J(x)$ (and thus also \bar{k}^2) is invariant under all the symmetries of the lattice; (3) the interaction is ferromagnetic, so that $\bar{k}^2 = 0$ only for $k=0$ in the Brillouin zone.

Besides the leading (universal) rotationally invariant critical behavior, we are interested in understanding the effects of the lattice structure on the two-point function and the recovery of rotational invariance. For this reason, our analysis must take into account the irrelevant operators, which break rotational invariance. It is natural to expand $\bar{k}^2(k)$ in multipoles by writing

$$\bar{k}^2(k) = \sum_{l=0}^{\infty} \sum_{p=1}^{p_l} e_{2l}^{(p)}(k^2) Q_{2l}^{(p)}(k). \quad (10)$$

Here the functions $Q_{2l}^{(p)}(k)$ are multipole combinations, which are invariant under the symmetries of the lattice. Their expressions can be obtained from the fully symmetric traceless tensors of rank $2l$, $T_{2l}^{\alpha_1 \dots \alpha_{2l}}(k)$ [9,10], by considering all the cubic-invariant combinations, which can be obtained by setting equal an even number of indices larger than or equal to four and then summing over them. Odd-rank terms are absent in the expansion (10) because of the parity symmetry $x \rightarrow -x$. Moreover, there is no rank-two term, i.e., $Q_2(k) = 0$, due to the discrete rotational symmetry of the lattice. The summation over p in Eq. (10) is due to the fact that, for given l , there are in general many multipole combinations that are cubic invariant [11]. For notational simplicity, we will suppress the explicit dependence on p in all the following formulas, but the reader should remember that it is understood in the notation.

Let us give the explicit expressions of $Q_{2l}^{(p)}(k)$ for the first few values of l . We set $Q_0(k) \equiv 1$. For $l=2$ there is only one invariant combination, i.e., $p_2=1$, which can be derived from

$$\begin{aligned} T_4^{\alpha\beta\gamma\delta}(k) &= k^{\alpha} k^{\beta} k^{\gamma} k^{\delta} - \frac{k^2}{(d+4)} \text{Perm}_{\alpha\beta\gamma\delta}(\delta^{\alpha\beta} k^{\gamma} k^{\delta}) \\ &+ \frac{(k^2)^2}{(d+2)(d+4)} \text{Perm}_{\alpha\beta\gamma\delta}(\delta^{\alpha\beta} \delta^{\gamma\delta}), \quad (11) \end{aligned}$$

where $\text{Perm}_{\alpha_1 \dots \alpha_n}(\dots)$ indicates the sum of the nontrivial permutations of its arguments. One then defines

$$Q_4(k) = \sum_{\mu} T_4^{\mu\mu\mu\mu} = k^4 - \frac{3}{d+2} (k^2)^2, \quad (12)$$

where the notation $k^n \equiv \sum_{\mu} k_{\mu}^n$ is used. For $l=3$, $p_3=1$ for all $d > 2$. From the rank-six tensor $T_6^{\mu\nu\alpha\beta\gamma\delta}(k)$ one finds

$$Q_6(k) = \sum_{\mu} T_6^{\mu\mu\mu\mu\mu\mu}(k) = k^6 - \frac{15k^2 k^4}{d+8} + \frac{30(k^2)^3}{(d+4)(d+8)}. \quad (13)$$

In $d=2$ it is easy to verify that $Q_6(k)=0$ so that $p_3=0$. For $l=4$ and $d > 3$ two different $Q_8^{(p)}(k)$ can be extracted from the corresponding tensor $T_8^{\alpha_1 \dots \alpha_8}$: $Q_8^{(1)}(k) = \sum_{\mu} T_8^{\mu\mu\mu\mu\mu\mu\mu\mu}$ and $Q_8^{(2)}(k) = \sum_{\mu\nu} T_8^{\mu\mu\mu\mu\nu\nu\nu\nu}$. When $d=2,3$ the two combinations are not independent. Indeed $Q_8^{(2)} = 2Q_8^{(1)}$ so that $p_4 = 1$. Higher values of l can be dealt with similarly.

In order to study the formal continuum limit of the Hamiltonian defined in Eq. (7), we expand $e_{2l}(k^2)$ in powers of k^2 . We write (the sum over different multipoles with the same value of l being understood in the notation)

$$\bar{k}^2(k) = \sum_{l=0}^{\infty} \sum_{m=0}^{\infty} e_{2l,m} (k^2)^m Q_{2l}(k), \quad (14)$$

where $e_{0,0} \equiv 0$ and $e_{0,1} \equiv 1$. Inserting back in Eq. (7) one sees that Eq. (14) represents an expansion in terms of the irrelevant operators

$$O_{2l,m}(x) = \vec{s}(x) \cdot \square^m Q_{2l}(\partial) \vec{s}(x), \quad (15)$$

where $\square \equiv \sum_{\mu} \partial_{\mu}^2$. The leading operator that breaks rotational invariance is the four-derivative term

$$O_4(x) \equiv O_{4,0}(x) = \vec{s}(x) \cdot Q_4(\partial) \vec{s}(x), \quad (16)$$

which has canonical dimensions $d+2$.

Let us now consider the Green's function

$$G(x; \beta) \equiv \langle \vec{s}_0 \cdot \vec{s}_x \rangle, \quad (17)$$

and its Fourier transform $\tilde{G}(k; \beta)$. We define a zero-momentum mass scale $M_G(\beta)$ by

$$M_G(\beta) \equiv \frac{1}{\xi_G(\beta)}, \quad (18)$$

where $\xi_G(\beta)$ is the second-moment correlation length

$$\xi_G^2(\beta) = \frac{1}{2d} \frac{\sum_x |x|^2 G(x; \beta)}{\sum_x G(x; \beta)}. \quad (19)$$

Since there is a one-to-one correspondence between $M_G(\beta)$ and β , one may consider $\tilde{G}(k; \beta)$ as a function of $M_G(\beta)$ instead of β . Indeed, for the purpose of studying the critical limit, it is natural to consider $\tilde{G}(k; \beta)$ as a function of k and M_G . In complete analogy to our discussion of $\tilde{J}(k)$, we analyze the behavior of $\tilde{G}(k, M_G)$ in terms of multipoles

[again a sum over different multipole combinations with the same value of l is understood, see Eq. (10)]:

$$\tilde{G}^{-1}(k, M_G) = \sum_{l=0}^{\infty} g_{2l}(y, M_G) Q_{2l}(k), \quad (20)$$

where $y = k^2/M_G^2$. Notice that $Q_{2l}(k) = Q_{2l}(k/M_G)M_G^{2l}$.

For the purpose of studying the universal properties of the critical limit of $G(x)$, in which $M_G \rightarrow 0$ keeping k/M_G fixed, it is important to understand the behavior of the functions $g_{2l}(y, M_G)$ when $M_G \rightarrow 0$. The naive limit does not exist. However, as long as the contributions to $\tilde{G}^{-1}(k, M_G)$ are originated by the insertion of individual (irrelevant) operators without any mixing among different operators with the same symmetry properties, one can apply standard results in renormalization theory. In this case one can establish some universal properties. For a generic choice of $\tilde{J}(k)$ this holds only for the functions $g_0(y, M_G)$ and $g_4(y, M_G)$. Indeed for higher values of l there are mixings among different operators that make the renormalization of the functions $g_{2l}(y, M_G)$ more complicated. Consider, for instance, the case $l=3$ in the large- N limit, where the operators have canonical dimensions. In this case terms proportional to $Q_4(k)$ are depressed as M_G^2 , while terms proportional to $Q_6(k)$ are depressed as M_G^4 . However, it is easy to see that the multipole decomposition of $Q_4(k)^2$, which is also depressed as M_G^4 , contains a term of the form $k^2 Q_6(k)$. This means that there are two operators contributing to $g_6(k, M_G)$, $O_{4,0}(x)^2$, and $O_{6,0}(x)$. An analogous argument applies to higher values of l . Notice that for the particular case of $l=3$ the mixing should disappear in the limit $y \rightarrow 0$: thus for $M_G \rightarrow 0$ $g_6(0, M_G)$ can be directly related to the renormalization properties of the operator $O_{6,0}(x)$.

For $l=0$ and $l=2$ standard results of renormalization theory show that, if $Z_{2l}(M_G) \equiv g_{2l}(0, M_G)$, the following limit exists:

$$\lim_{M_G \rightarrow 0} \frac{g_{2l}(y, M_G)}{Z_{2l}(M_G)} = \hat{g}_{2l}(y), \quad (21)$$

where $\hat{g}_{2l}(y)$ is a smooth function, which is normalized so that $\hat{g}_{2l}(0) = 1$. The function $\hat{g}_{2l}(y)$ is universal in the sense that it is independent of the specific Hamiltonian.

The function $\hat{g}_4(y)$ can also be obtained by considering the linear response of the system to an external field possessing the corresponding symmetry properties. One considers the one-particle irreducible two-point function with an insertion of a $O_{2l,0}(x)$ operator at zero momentum, i.e.,

$$\Gamma_{O_{2l}}(x, M_G) \equiv \int dz \langle O_{2l,0}(z) \vec{s}(0) \cdot \vec{s}(x) \rangle^{\text{irr}} \quad (22)$$

and the corresponding Fourier transform $\tilde{\Gamma}_{O_{2l}}(k, M_G)$. Setting

$$\bar{Z}_{2l}(M_G) \equiv \lim_{k \rightarrow 0} \frac{\tilde{\Gamma}_{O_{2l}}(k, M_G)}{Q_{2l}(k)}, \quad (23)$$

the following limit exists

$$\lim_{M_G \rightarrow 0} \frac{\tilde{\Gamma}_{O_{2l}}(k, M_G)}{\bar{Z}_{2l}(M_G)} = \hat{g}_{2l}(y) Q_{2l}(k). \quad (24)$$

For $l=2$ the function defined by the previous equation coincides with that defined in Eq. (21); moreover for $M_G \rightarrow 0$, $Z_4(M_G)/\bar{Z}_4(M_G)$ is a finite (nonuniversal) constant, meaning that both quantities have the same singular behavior for $M_G \rightarrow 0$. For higher values of l , formula (24) still holds, but there is no easy relation between $\hat{g}_{2l}(y)$ and $g_{2l}(y, M_G)$ as defined in Eq. (20), at least for generic Hamiltonians. Indeed, at least in principle, one may consider specific forms of $\tilde{J}(k)$ enjoying the property that all contributions $g_{2n}(k, M_G)$ with $0 < n < \bar{l}$ vanish in the critical limit, for a given value of \bar{l} . In lattice quantum field theory this is essentially the spirit of Symanzik's improvement program [14]. In this case formula (21) is valid for $l = \bar{l}$ and the corresponding function $\hat{g}_{2l}(y)$ coincides with that defined by Eq. (24).

The functions $\hat{g}_{2l}(y)$ defined in Eq. (24) have a regular expansion in y around $y=0$:

$$\hat{g}_{2l}(y) = 1 + c_{2l,1}y + c_{2l,2}y^2 + \dots \quad (25)$$

$c_{0,1} = 1$ due to the definition of the second-moment correlation length.

The renormalization constant $\bar{Z}_{2l}(M_G)$ is instead nonuniversal. For $M_G \rightarrow 0$ it behaves as

$$\bar{Z}_{2l}(M_G) \approx \bar{z}_{2l} M_G^{-\eta_{2l}}, \quad (26)$$

where η_{2l} is a critical exponent that depends only on the spin of the representation [i.e., it does not depend on the additional index p which has always been understood in the notation, see Eq. (10)], and \bar{z}_{2l} is a nonuniversal constant that depends on the lattice and on the Hamiltonian (and the additional index p). An analogous expression is valid for $Z_4(M_G)$ (and for $Z_{2\bar{l}}$ for the special Hamiltonians we have discussed before): for $M_G \rightarrow 0$ we have $Z_4(M_G) \approx z_4 M_G^{-\eta_4}$. For $l=0$, as a consequence of our definitions, $Z_0(M_G) \sim M_G^{2-\eta}$, where η is the standard anomalous dimension of the field. More generally $\sigma_{2l} \equiv \eta - \eta_{2l}$ is the anomalous dimension of the irrelevant operator $O_{2l,0}(x)$.

In two dimensions and for $N \geq 3$ the renormalization constants diverge only logarithmically and thus we write for $l \neq 0$

$$\bar{Z}_{2l}(M_G) \approx \bar{z}_{2l} (\ln M_G)^{\gamma_{2l}} \left[1 + O\left(\frac{1}{\ln M_G}\right) \right]. \quad (27)$$

The anomalous dimensions γ_{2l} are universal while the prefactor \bar{z}_{2l} depends on the details of the interaction.

We can now discuss the critical limit of Eq. (20). Using the previous formulas we can write for $M_G \rightarrow 0$

$$\begin{aligned} \tilde{G}^{-1}(k, M_G) &\approx \hat{g}_0(y) + \text{r.i.s.} + \frac{z_4}{z_0} M_G^{2+\eta-\eta_4} \hat{g}_4(y) Q_4(k/M_G) \\ &+ \dots, \end{aligned} \quad (28)$$

where r.i.s. indicates rotationally invariant subleading corrections and the dots stand for terms that vanish faster as $M_G \rightarrow 0$. From Eq. (28) one immediately convinces oneself that the anisotropic effects in $G(x)$ vanish for $M_G \rightarrow 0$ as M_G^ρ where ρ is a universal critical exponent given by

$$\rho = 2 + \eta - \eta_4. \quad (29)$$

We must notice that the exponent ρ is not related to the exponent ω , which characterizes the critical behavior of the ‘‘rotationally invariant subleading’’ terms that vanish as M_G^ω , as they are connected to different (rotationally invariant) irrelevant operators. Finally notice that the leading term breaking rotational invariance is universal apart from a multiplicative constant, the factor z_4/z_0 .

Let us now consider the small-momentum limit in which $y \rightarrow 0$, keeping M_G fixed. In this case one can write for $l = 0, 2$ (or in the special case we have discussed above for $l = 0, \bar{1}$)

$$g_{2l}(y, M_G) = \sum_{m=0}^{\infty} u_{2l,m}(M_G) y^m. \quad (30)$$

By comparing this expansion with Eq. (25) and using Eq. (21), one recognizes that

$$Z_{2l}(M_G) = u_{2l,0}(M_G), \quad (31)$$

and

$$c_{2l,m} = \lim_{M_G \rightarrow 0} \frac{u_{2l,m}(M_G)}{u_{2l,0}(M_G)}. \quad (32)$$

In the following sections we will use this formula to derive estimates for $c_{2l,m}$. Indeed the functions $u_{2l,m}(M_G)$ can be determined by computing dimensionless invariant ratios of moments of $G(x; \beta)$:

$$q_{2l,m}(\beta) = \sum_x (x^2)^m Q_{2l}(x) G(x; \beta). \quad (33)$$

It is interesting to notice that the expansion (28) implies some universality properties for some ratios of $q_{2l,m}$. It is easy to verify that

$$R_{4,m,n}(\beta) = \frac{q_{0,n}(\beta) q_{4,m}(\beta)}{q_{0,m}(\beta) q_{4,n}(\beta)} \quad (34)$$

is universal for $T \rightarrow T_c$; indeed the constant z_4/z_0 drops out in the ratio. Notice that this means not only that $R_{4,m,n}$ does not depend on the particular Hamiltonian, but also that it is independent of the lattice structure as long as $O_{4,0}(x)$ is the leading operator breaking rotational invariance.

B. Other regular lattices

All the considerations of the previous subsection can be extended without changes to other lattices with cubic symmetry, such as the bcc and the fcc lattices. For other Bravais lattices the same general formulas hold, but different multipole combinations will appear in the expansion, according to the symmetry of the lattice. In general a larger number of

multipole combinations with given spin appear when considering lattices with a lower symmetry. It is important to notice that in order to have a rotationally invariant critical limit no multipole $Q_l(k)$ with $l=2$ should appear in the expansion of the Hamiltonian. Thus our considerations apply directly only to highly symmetric lattices with a tetrahedral or larger discrete rotational group. Indeed, if the term associated with $Q_2(k)$ appears in the multipole expansion of the Hamiltonian, and therefore of $\tilde{G}^{-1}(k, M_G)$, the critical limit is not rotationally invariant. However, it is always possible to eliminate such terms with an anisotropic change of the length scales [12,13]. Thus one can apply our analysis also to this case, provided one changes appropriately the physical meaning of $Q_4(k)$.

As an example of a non-cubic-symmetric lattice let us consider the two-dimensional triangular lattice. It is invariant under rotation of $\pi/3$. The relevant multipoles are

$$T_{6l}(k) = (-k^2)^{3l} \cos(6l\theta) = \sum_{m=0}^{3l} \binom{6l}{2m} k_y^{2m} (ik_x)^{6l-2m}, \quad (35)$$

where we have set $k_x = |k| \cos \theta$, $k_y = |k| \sin \theta$, and we have assumed one of the generators of the lattice to be parallel to the x axis. Thus in this case we write

$$\bar{k}^2(k) = \sum_{l=0}^{\infty} T_{6l}(k) e_{6l}(k^2), \quad (36)$$

and a similar expression for the expansion of the two-point function. For the triangular lattice the first operator that breaks rotational invariance has dimension $d+4$. This is a consequence of the fact that the triangular lattice has a larger symmetry group with respect to the square lattice. We define moments corresponding to $T_{6l}(k)$ by

$$t_{6l,m}(\beta) = \sum_x (x^2)^m T_{6l}(x) G(x; \beta). \quad (37)$$

The arguments given in the previous subsection can be generalized to the triangular lattice in a straightforward way. One derives an expansion of the form (28) with $\rho = 4 + \eta - \eta_6$, $T_6(k/M_G)$ and $\hat{g}_6(y)$ substituting $Q_4(k/M_G)$, and $\hat{g}_4(y)$.

C. Non-Bravais lattices

Up to now we have considered regular (Bravais) lattices. However, other important lattice structures are represented by lattices with basis. Particular examples are the honeycomb lattice in two dimensions and the diamond lattice in three dimensions. These lattices are generically defined by the set of points \vec{x} such that $\vec{x} = \vec{x}' + p \vec{\eta}_p$ and $\vec{x}' = \sum_i l_i \vec{\eta}_i$, where $\vec{\eta}_p$ is the so-called basis vector joining the two points of the basis, and $\vec{\eta}_i$ are the generators of the underlying regular lattice. Here $p=0,1$ and $l_i \in \mathbb{Z}$. For the honeycomb lattice $\vec{\eta}_i$ are the generators of a triangular lattice while for the diamond lattice $\vec{\eta}_i$ are the generators of a fcc lattice. Due to the breaking of translational invariance one distinguishes between correlations between points with the same value of

p (i.e., points belonging to the same regular lattice) and points with different p . In general the components $G_{pp'}$ of the two-point correlation function can be written in the form

$$G_{00}(x-y) = G_{11}(x-y) = \int \frac{dk}{V_B} e^{ik(x-y)} \frac{1}{\Delta(k, M_G)} \quad (38)$$

and

$$G_{01}(x-y) = G_{10}(y-x) = \int \frac{dk}{V_B} e^{ik(x-y)} \frac{H(k, M_G)}{\Delta(k, M_G)}, \quad (39)$$

where the integrals are performed over the Brillouin zone of the corresponding underlying regular lattice, V_B being its volume. $G_{11}(x)$ and therefore $\Delta(k, M_G)$ have the symmetries of the underlying regular lattice and thus can be expanded as in the first subsection. On the other hand, $H(k, M_G)$ does not have the symmetry of the regular lattice, but only the reduced symmetry of the full lattice. For the Gaussian model with nearest-neighbor interactions defined on the honeycomb and diamond lattices (and also on their d -dimensional generalization), it is easy to realize that, when $M_G \rightarrow 0$,

$$\Delta(k, M_G) \rightarrow d[1 - |H(k, 0)|^2] + M_G^2, \quad (40)$$

and $\Delta(k, M_G)$ turns out to be the inverse propagator for the Gaussian theory defined on the corresponding regular lattice.

Because of the reduced symmetry, additional multipoles that are not parity invariant appear in the expansion of $H(k, M_G)$. In the case of the honeycomb lattice the symmetry of the triangular lattice is reduced to $\theta \rightarrow \theta + 2\pi/3$. Assuming that one of the links leaving a site is parallel to the x axis, one can write

$$H(k, M_G) = \sum_{l=0}^{\infty} T_{3l}(k) h_{3l}(y, M_G), \quad (41)$$

where we have extended the definition (35) to include odd multipoles:

$$T_{3l}(k) = (-k^2)^{3l/2} \cos(3l\theta) = \sum_{m=0}^{3l/2} \binom{3l}{2m} k_y^{2m} (ik_x)^{3l-2m}. \quad (42)$$

The factor i in this equation ensures that the functions $h_{3l}(y, M_G)$ are real for all l .

For the diamond lattice one can write

$$H(k, M_G) = \sum_{l=0}^{\infty} \sum_{p=1}^{p_l} Q_l^{(p)}(k) h_l^{(p)}(y, M_G), \quad (43)$$

where $Q_l^{(p)}(k)$ are multipoles constructed from $T_l^{\alpha_1 \dots \alpha_l}$ as in the case of the cubic lattices. The only difference is that now odd-spin operators are allowed, belonging to the class

$$Q_{2l+3}(k) = ik_1 k_2 k_3 Q_{2l}(k), \quad (44)$$

where we have assumed the natural orientation of the underlying fcc lattice.

For these lattices, it is not straightforward to make contact with the field-theoretical approach. The problem is writing

down operators in the effective Hamiltonian that break the parity symmetry. These operators must have an odd number of derivatives, but, if they are bilinear in a real field ϕ , they give after integration only boundary terms. The solution to this apparent puzzle comes from the fact that the effective Hamiltonian for models on lattices with basis is naturally written down in terms of two fields, defined on the two regular sublattices.

As in the regular lattice case, we can associate to the breaking of the parity symmetry a universal exponent ρ_p . In principle it can be derived from the critical dimension of the lower-dimensional operator breaking this symmetry. From a practical point of view it is simpler to consider moments of $G(x)$. For the diamond lattice one defines ρ_p from the behavior, for $M_G \rightarrow 0$, of the odd moments $q_{3,m}(\beta)$, i.e.,

$$\frac{q_{3,m}(\beta)}{q_{0,0}(\beta)} \sim M_G^{-3-2m+\rho_p}. \quad (45)$$

The same formula applies to the honeycomb lattice with the obvious substitutions, $q_{0,0} \rightarrow t_{0,0}$, $q_{3,m} \rightarrow t_{3,m}$.

III. CRITICAL BEHAVIOR OF $G(x)$ AT LOW MOMENTUM

A. Parametrization of the spherical limit of $G(x)$ at low momentum

According to the discussion presented in the previous section, in the critical limit multipole contributions are depressed by powers of M_G , hence for $\beta \rightarrow \beta_c$,

$$\frac{\tilde{G}(0; \beta)}{\tilde{G}(k; \beta)} \rightarrow \hat{g}_0(y), \quad (46)$$

where, again, $y = k^2/M_G^2$. As stated in Eq. (25), $\hat{g}_0(y)$ can be expanded in powers of y around $y=0$:

$$\hat{g}_0(y) = 1 + y + \sum_{i=2}^{\infty} c_i y^i, \quad (47)$$

where $c_i \equiv c_{0,i}$. For generalized Gaussian theories $c_i = 0$. As discussed in Sec. II A the coefficients c_i of the low-momentum expansion of $\hat{g}_0(y)$ can be related to the critical limit of appropriate dimensionless ratios of spherical moments $m_{2j} \equiv q_{0,j}$ or of the corresponding weighted moments

$$\bar{m}_{2j} \equiv \frac{m_{2j}}{m_0}. \quad (48)$$

Introducing the quantities

$$v_{2j} = \frac{1}{2^j j! \prod_{i=0}^{j-1} (d+2i)} \bar{m}_{2j} M_G^{2j}, \quad (49)$$

one may compute $\hat{u}_i \equiv u_{0,i}/u_{0,0}$ [cf. Eq. (30)] from the following combinations of v_{2j} ,

$$\hat{u}_2 = 1 - v_4, \quad \hat{u}_3 = 1 - 2v_4 + v_6, \quad (50)$$

etc. By definition, see Eqs. (32) and (47), in the critical limit $\hat{u}_i \rightarrow c_i$.

Another important quantity that characterizes the low-momentum behavior of $\hat{g}_0(y)$ is the critical limit of the ratio M^2/M_G^2 ,

$$S_M \equiv \lim_{\beta \rightarrow \beta_c} \frac{M^2}{M_G^2}, \quad (51)$$

where M is the mass gap of the theory, that is the mass determining the long-distance exponential behavior of $G(x)$. The value of S_M is related to the negative zero y_0 of $\hat{g}_0(y)$, which is closest to the origin by $y_0 = -S_M$. The constant S_M is one in Gaussian models [i.e., when $\hat{g}_0(y) = 1 + y$], as the large- N limit of $O(N)$ models.

Let us now consider the relation between the zero-momentum renormalization constant

$$Z_G \equiv \chi M_G^2 = Z_0^{-1} M_G^2, \quad (52)$$

where Z_0 has been introduced in Eq. (21), and the on-shell renormalization constant Z , which is defined by

$$\tilde{G}(k) \rightarrow \frac{Z}{M^2 + k^2} \quad (53)$$

when $k \rightarrow iM$. The mass gap M and the constant Z determine the large-distance behavior of $G(x)$; indeed for $|x| \rightarrow \infty$,

$$G(x) \rightarrow \frac{Z}{2M} \left(\frac{M}{2\pi|x|} \right)^{(d-1)/2} e^{-M|x|}. \quad (54)$$

The critical limit S_Z of the ratio Z_G/Z is a universal quantity given by

$$S_Z = \lim_{\beta \rightarrow \beta_c} \frac{Z_G}{Z} = \frac{\partial}{\partial y} \hat{g}_0(y) \Big|_{y=y_0}. \quad (55)$$

In a Gaussian theory $Z_G = Z$.

B. $1/N$ expansion

In the large- N limit the difference $\hat{g}_0(y) - (1 + y)$ is depressed by a factor $1/N$. It can be derived from the $1/N$ expansion of the self-energy in the continuum formulation. One finds [5,15]

$$\hat{g}_0(y) = 1 + y + \frac{1}{N} \phi_1(y) + O\left(\frac{1}{N^2}\right), \quad (56)$$

where, for $d=3$,

$$\phi_1(y) = \frac{2}{\pi} \int_0^\infty dz \frac{z}{\arctan(\frac{1}{2}\sqrt{z})} \left[\frac{1}{4\sqrt{yz}} \ln \left(\frac{y+z+2\sqrt{yz}+1}{y+z-2\sqrt{yz}+1} \right) - \frac{1}{z+1} + \frac{y(3-z)}{3(z+1)^3} \right]. \quad (57)$$

A general discussion of the $O(1/N)$ correction to $\hat{g}_0(y)$ in d dimensions is presented in Appendix A. The coefficients c_i of the low-momentum expansion of $\hat{g}_0(y)$ turn out to be very small. Writing

$$c_i = \frac{c_i^{(1)}}{N} + O\left(\frac{1}{N^2}\right), \quad (58)$$

one obtains $c_2^{(1)} = -0.004\,448\,60$, $c_3^{(1)} = 0.000\,134\,410$, $c_4^{(1)} = -0.000\,006\,5805$, $c_5^{(1)} = 0.000\,000\,4003$, etc. We have computed $c_i^{(1)}$ up to $i=25$; a straightforward application of the ratio method indicates that the convergence radius of the series $\sum_i c_i^{(1)} y^i$ is $y_r \approx 9$. This is expected since the singularity closest to the origin should be the three-particle cut y_{cut} . Assuming that no three-particle bound states exist, then $y_{\text{cut}} = -9S_M \approx -9$, in agreement with our findings.

For sufficiently large N we then expect

$$c_i \ll c_2 \ll 1 \quad \text{for } i \geq 3. \quad (59)$$

As we shall see from the analysis of the strong-coupling expansion of $G(x)$, the pattern (59) is verified also at low values of N .

We have also computed S_M and S_Z to $O(1/N)$. Writing

$$S_\# = 1 + \frac{S_\#^{(1)}}{N} + O\left(\frac{1}{N^2}\right), \quad (60)$$

one finds $S_M^{(1)} = \phi_1(-1) = -0.004\,590\,02$, and $S_Z^{(1)} = \phi_1'(-1) = 0.009\,328\,94$.

As expected from the relations (59) among the coefficients c_i , a comparison with Eq. (58) shows that the non-Gaussian corrections to S_M and S_Z are essentially determined by the term proportional to $(k^2)^2$ in $\tilde{G}^{-1}(k)$, through the approximate relations

$$S_M \approx 1 + c_2, \quad (61)$$

$$S_Z \approx 1 - 2c_2, \quad (62)$$

with corrections of $O(c_3)$.

C. g expansion in three dimensions

Another approach to the study of the critical behavior in the symmetric phase of $O(N)$ models is based on the so-

called g expansion, the perturbative expansion at fixed dimension $d=3$ for the corresponding ϕ^4 continuum formulation [16]. The perturbative series that are obtained in this way are asymptotic; nonetheless accurate results can be obtained using a Borel transformation and a conformal mapping, which take into account their large-order behavior. As general references on this method see, for instance, Refs. [2] and [17]. This technique has led to very precise estimates of the critical exponents.

Starting from the continuum action (6), one renormalizes the theory at zero momentum using the following renormalization conditions for the irreducible two- and four-point correlation functions of the field ϕ :

$$\Gamma^{(2)}(p)_{\alpha\beta} = Z_G^{-1} \Gamma_R^{(2)}(p) \delta_{\alpha\beta}, \quad (63)$$

$$\Gamma^{(4)}(0,0,0,0)_{\alpha\beta\gamma\delta} = + Z_G^{-2} \frac{g}{3} M_G \delta_{\alpha\beta\gamma\delta}, \quad (64)$$

where $\Gamma_R^{(2)}(p) = M_G^2 + p^2 + O(p^4)$, and $\delta_{\alpha\beta\gamma\delta} \equiv \delta_{\alpha\beta} \delta_{\gamma\delta} + \delta_{\alpha\gamma} \delta_{\beta\delta} + \delta_{\alpha\delta} \delta_{\beta\gamma}$. When $M_G \rightarrow 0$ the renormalized coupling constant is driven toward an infrared stable zero g^* of the β function $\beta(g) \equiv M_G \partial g / \partial M_G|_{g_0, \Lambda}$.

The universal function $\hat{g}_0(y)$ is related to the renormalized function $f(g, y) \equiv M_G^{-2} \Gamma_R^{(2)}(k)$ by $\hat{g}_0(y) = f(g^*, y)$. We computed the first three nontrivial orders of the non-Gaussian corrections to $\hat{g}_0(y)$. A calculation up to four loops gave

$$f(g, y) = 1 + y + \bar{g}^{-2} Z_g^2 Z_G \frac{N+2}{(N+8)^2} \varphi_2(y) + \bar{g}^{-3} Z_g^3 Z_G^{3/2} \frac{N+2}{(N+8)^2} \varphi_3(y) + \bar{g}^{-4} Z_g^4 Z_G^2 \frac{N+2}{(N+8)^2} \left[\frac{(N+2)}{(N+8)^2} \varphi_{4,1}(y) + \frac{(N^2+6N+20)}{(N+8)^2} \varphi_{4,2}(y) + \frac{(5N+22)}{(N+8)^2} \varphi_{4,3}(y) \right] + O(\bar{g}^{-5}), \quad (65)$$

where \bar{g} is the rescaled coupling [2] $\bar{g} = (N+8)g/48\pi$, Z_g is the renormalization constant of the coupling (defined by $g_0 = M_G g Z_g$)

$$Z_g = 1 + \bar{g} + \left[1 - \frac{2(41N+190)}{27(N+8)^2} \right] \bar{g}^2 + O(\bar{g}^3), \quad (66)$$

and Z_G is the zero-momentum renormalization of the field

$$Z_G = 1 - \frac{4(N+2)}{27(N+8)^2} \bar{g}^2 + O(\bar{g}^3). \quad (67)$$

A simple derivation of the two- and three-loop functions $\varphi_2(y)$ and $\varphi_3(y)$ is presented in Appendix A [cf. Eqs. (A14)]. In particular using the results of Refs. [18,19] one finds

$$\varphi_2(y) = 4 \ln(1 + \frac{1}{9}y) + 24 \frac{\arctan(\sqrt{y}/3)}{\sqrt{y}} - 8 - \frac{4}{27}y. \quad (68)$$

We shall not report the expressions of the four-loop functions $\varphi_{4,j}(y)$ because they are not very illuminating.

The coefficients of the low-momentum expansion can be easily obtained from Eq. (66) by calculating the zero-momentum derivatives of the functions $\varphi_{n,j}(y)$. We write

$$c_i = \frac{N+2}{(N+8)^2} h_i^{(2)} \bar{g}^2 + \frac{N+2}{(N+8)^2} (2h_i^{(2)} + h_i^{(3)}) \bar{g}^3 + \frac{N+2}{(N+8)^2} \left\{ 2h_i^{(2)} \left[1 - \frac{8(7N+32)}{3(N+8)^2} \right] + 3h_i^{(3)} + h_i^{(4,1)} \frac{(N+2)}{(N+8)^2} + h_i^{(4,2)} \frac{(N^2+6N+20)}{(N+8)^2} + h_i^{(4,3)} \frac{(5N+22)}{(N+8)^2} \right\} \bar{g}^4 + O(\bar{g}^5), \quad (69)$$

where we have introduced the coefficients

$$h_i^{(n,j)} = \frac{1}{i!} \frac{d^i}{dy^i} \varphi_{n,j}(y) \Big|_{y=0}. \quad (70)$$

In Table I we report the numerical values of $h_i^{(k,j)}$ for $i \leq 5$. The calculation of S_M and S_Z to $O(\bar{g}^3)$ requires the values

of the functions $\varphi_2(y)$ and $\varphi_3(y)$ and of their derivatives at $y = -1$: $\varphi_2(-1) = -0.005\,217\,83$, $2\varphi_2(-1) + \varphi_3(-1) = -0.000\,282\,71$, $\varphi_2'(-1) = 0.010\,7349$, and $2\varphi_2'(-1) + \varphi_3'(-1) = 0.000\,4490$.

A comparison of the g expansions of c_i , S_M , and S_Z shows that the approximate relations (61) and (62) are valid for all values of N and not only for $N \rightarrow \infty$ as shown in the previous subsection.

TABLE I. Numerical values of the coefficients $h_i^{(n,j)}$ defined in Eq. (70).

i	$h_i^{(2)}$	$h_i^{(3)}$	$h_i^{(4,1)}$	$h_i^{(4,2)}$	$h_i^{(4,3)}$
2	$-\frac{2}{405}$	0.00949125	0.000765804	-0.0134856	-0.0345992
3	$\frac{4}{15309}$	-0.000612784	-0.00000341189	0.00102554	0.00253490
4	$-\frac{1}{59049}$	0.0000450060	-0.00000233206	-0.0000841861	-0.00020327
5	$\frac{4}{3247695}$	-0.0000035762	0.00000035769	0.0000072651	0.00002390

In order to get quantitative estimates, one must perform a resummation of the series and then evaluate it at the fixed-point value of the coupling \bar{g}^* . Although the terms of the g expansion we have calculated are only three for c_i and two for S_M and S_Z , we have tried to extract quantitative estimates that take into account also the following facts:

(i) The g expansion is Borel summable [20] (see also, e.g., Refs. [2] and [17] for a discussion of this issue), and the singularity closest to the origin of the Borel transform (corresponding to the rescaled coupling \bar{g}) is known [21]: $b_s = -0.751\,897\,74 \times (N+8)$.

(ii) The fixed point value \bar{g}^* of \bar{g} has been accurately determined by analyzing a much longer expansion [to $O(g^7)$] of the corresponding β function [22–25]. Independent and consistent estimates of \bar{g}^* have been obtained by other approaches, such as strong-coupling expansion of lattice N -vector model [26,27] (for $N=1$ see also Refs. [28–31]), and Monte Carlo lattice simulations (only data for $N=1$ are available [32–35]).

We have followed the procedure described in Ref. [36] (see also Ref. [2]), where the perturbative expansion in powers of \bar{g} is summed using a Borel transformation and a conformal mapping, which takes into account its large-order behavior. Since the g series of c_i , S_M-1 , and S_Z-1 have the form $R(\bar{g}) = \bar{g}^2 \sum_{k=0} a_k \bar{g}^k$, one may apply the resummation method either to $R(\bar{g})$ or to $R(\bar{g})/\bar{g}^2$. In Table III we present results for both choices. In our calculations we used the estimates of \bar{g}^* obtained from the analysis of the β function by [22,23,25]. They are reported in Table II. For small values of N slightly lower values of \bar{g}^* were computed in Ref. [37], taking into account the possible nonana-

lyticity of the β function at the critical point [24]. This difference is, however, too small to be quantitatively relevant in our calculations.

It is difficult to estimate the uncertainty of the results. Resummations of $R(\bar{g})$ are not very stable and indeed the estimates show an upward trend with the order of the series: roughly we expect an error $\lesssim 20\%$ on c_i and S_M for small values of N . As N increases the estimates become more precise. Resummations of $R(\bar{g})/\bar{g}^2$ appear instead much more stable: results with two terms essentially agree with the final estimates using three terms. In this case the error should be $\lesssim 5\%$ on c_i and S_M for small values of N and again it decreases as N increases. The final results are in good agreement with the estimates by other methods.

D. ϵ expansion

The universal function $\hat{g}_0(y)$ can be computed perturbatively in $\epsilon=4-d$ using the continuum ϕ^4 theory [38]. The leading order is simply $\hat{g}_0(y)=1+y$. The first correction appears at order ϵ^2 and was computed by Fisher and Aharony [4]. We have extended the series, calculating the $O(\epsilon^3)$ term, obtaining

$$\hat{g}_0(y) = 1 + y + \epsilon^2 \frac{N+2}{(N+8)^2} \left[1 + \epsilon \frac{6(3N+14)}{(N+8)^2} \right] \psi_2(y) + \epsilon^3 \frac{N+2}{(N+8)^2} \psi_3(y) + O(\epsilon^4), \quad (71)$$

where

TABLE II. For several values of N and for the cubic and diamond lattice, we report the values of β_c we used in our strong-coupling calculations. We also report the fixed-point value \bar{g}^* of the rescaled zero-momentum four-point renormalized coupling, as obtained by field-theoretical methods.

N	Cubic	Diamond	\bar{g}^*
0	$\beta_c = 0.213492(1)$ [50]	$\beta_c = 0.34737(1)$ [45]	1.421(8) [2]
1	$\beta_c = 0.2216544(3)$ [51]	$\beta_c = 0.3697(1)$	1.416(5) [2]
2	$\beta_c = 0.22710(1)$ [52]	$\beta_c = 0.3845(2)$	1.406(4) [2]
3	$\beta_c = 0.231012(12)$ [53]	$\beta_c = 0.3951(2)$	1.391(4) [2]
4	$\beta_c = 0.23398(2)$ [43]	$\beta_c = 0.4027(2)$	1.369 [25]
8	$\beta_c = 0.24084(3)$ [43]	$\beta_c = 0.4200(2)$	1.303 [25]
16	$\beta_c = 0.24587(6)$	$\beta_c = 0.4327(2)$	1.207 [25]
32	$\beta_c = 0.2491(1)$	$\beta_c = 0.4401(1)$	1.122 [25]
∞	$\beta_c = 0.252731 \dots$ [54]	$\beta_c = 0.448220 \dots$	1

TABLE III. Estimates of c_2 , c_3 , S_M , and d_1 as obtained by the analysis of the strong-coupling expansion of $G(x)$ on the cubic and diamond lattice, from resummations of the available g expansion and ϵ expansion [in this case we give two numbers corresponding to the two choices: resumming $R(x)$ or $R(x)/x^2$], and from the $O(1/N)$ calculation of $1/N$ expansion.

N		$10^4 c_2$	$10^4 c_3$	$10^4(S_M - 1)$	$10^4 d_1$
0	Cubic	-1(1)	0.12(1)	0(2)	1(1)
	Diamond	-1(1)	0.10(1)	0(1)	-1.0(5)
	g expansion	-3.29, -3.63	0.108, 0.102	-2.95, -3.50	-1.31, -1.60
	ϵ expansion	-2.48, -4.26	0.065, 0.114	-2.55, -4.38	-1.10
1	Cubic	-3.0(2)	0.10(1)	-2.5(1.0)	-1.7(5)
	Diamond	-3.0(2)	0.10(2)	-2.3(4)	-3(1)
	g expansion	-3.92, -4.27	0.126, 0.120	-3.50, -4.12	-1.59, -1.89
	ϵ expansion	-3.06, -4.99	0.080, 0.134	-3.14, -5.13	-1.31
2	Impr ϵ expansion	-2.80, -3.64	0.060, 0.089	-2.86, -3.73	
	Cubic	-3.9(2)	0.11(1)	-3.5(1.0)	-2.3(2)
	Diamond	-4.1(4)	0.10(2)	-3.5(3)	-3(1)
	g expansion	-4.22, -4.54	0.133, 0.128	-3.85, -4.40	-1.72, -2.01
3	ϵ expansion	-3.39, -5.29	0.089, 0.142	-3.48, -5.44	-1.41
	Cubic	-4.1(1)	0.11(2)	-4.1(4)	-2.5(2)
	Diamond	-4.5(3)	0.11(3)	-4.0(4)	-2.6(3)
	g expansion	-4.29, -4.58	0.134, 0.128	-3.96, -4.45	-1.77, -2.03
4	ϵ expansion	-3.56, -4.55	0.094, 0.144	-3.66, -5.50	-1.46
	Cubic	-4.1(2)	0.12(1)	-4(1)	-2.5(2)
	Diamond	-4.7(2)	0.10(2)	-4.2(4)	-2.5(5)
	g expansion	-4.21, -4.46	0.130, 0.125	-3.92, -4.34	-1.76, -1.99
8	ϵ expansion	-3.64, -5.28	0.096, 0.143	-3.74, -5.43	-1.47
	$1/N$ expansion	-11.12	0.336	-11.48	-5.12
	Cubic	-3.5(1)	0.09(2)	-3.8(5)	-2.1(2)
	Diamond	-4.0(1)	0.05(5)	-3.8(4)	-3(2)
16	g expansion	-3.60, -3.72	0.108, 0.103	-3.44, -3.68	-1.55, -1.68
	ϵ expansion	-3.48, -4.55	0.093, 0.124	-3.58, -4.68	-1.38
	$1/N$ expansion	-5.56	0.118	-5.74	-2.56
	Cubic	-2.4(1)	0.06(1)	-2.8(2)	-1.4(2)
32	Diamond	-2.65(5)	0.05(3)	-2.7(3)	-1.2(8)
	g expansion	-2.46, -2.49	0.072, 0.069	-2.43, -2.52	-1.10, -1.15
	ϵ expansion	-2.73, -3.19	0.074, 0.088	-2.81, -3.28	-1.10
	$1/N$ expansion	-2.78	0.084	-2.87	-1.28
∞	Cubic	-1.45(5)	0.04(1)	-1.8(3)	-0.7(2)
	Diamond	-1.50(5)	0.04(1)	-1.7(3)	-0.5(3)
	g expansion	-1.427, -1.429	0.041, 0.040	-1.45, -1.48	-0.66, -0.67
	ϵ expansion	-1.73, -1.84	0.047, 0.052	-1.78, -1.90	-0.75
$1/N$ expansion	-1.39	0.042	-1.43	-0.64	
∞		0	0	0	

$$\psi_2(y) = 2 \int_0^\infty \sqrt{z(1+\frac{1}{4}z)} \ln(\sqrt{1+\frac{1}{4}z} + \frac{1}{2}\sqrt{z}) h(y,z), \quad (72)$$

$$h(y,z) = -\frac{1}{1+z} + \frac{y}{(1+z)^3} + \frac{1}{2yz}(1+y+z - \sqrt{1+2y+2z+y^2-2yz+z^2}).$$

We do not report the explicit expression of $\psi_3(y)$ because it is not very illuminating. It can, however, be obtained from Eqs. (A10), (A11), and (A13) of Appendix A, where we show how to derive the functions $\psi_2(y)$ and $\psi_3(y)$ from the $O(1/N)$ calculation of $\hat{g}_0(y)$ in d dimensions.

The coefficients c_i to $O(\epsilon^3)$ can be derived from Eq. (71) and the expansions of $\psi_2(y)$ and $\psi_3(y)$ around $y=0$:

$$\begin{aligned} \psi_2(y) = & -7.52024 \times 10^{-3} y^2 + 1.91931 \times 10^{-4} y^3 \\ & - 8.14201 \times 10^{-6} y^4 + 4.39145 \times 10^{-7} y^5 + O(y^6), \end{aligned} \quad (73)$$

$$\begin{aligned} \psi_3(y) = & 1.87481 \times 10^{-3} y^2 - 2.50674 \times 10^{-5} y^3 \\ & + 2.48598 \times 10^{-8} y^4 + 5.15004 \times 10^{-8} y^5 + O(y^6). \end{aligned} \quad (74)$$

The calculation of S_M and S_Z to $O(\epsilon^3)$ requires the values of $\psi_2(y)$ and $\psi_3(y)$ and of their derivatives at $y=-1$:

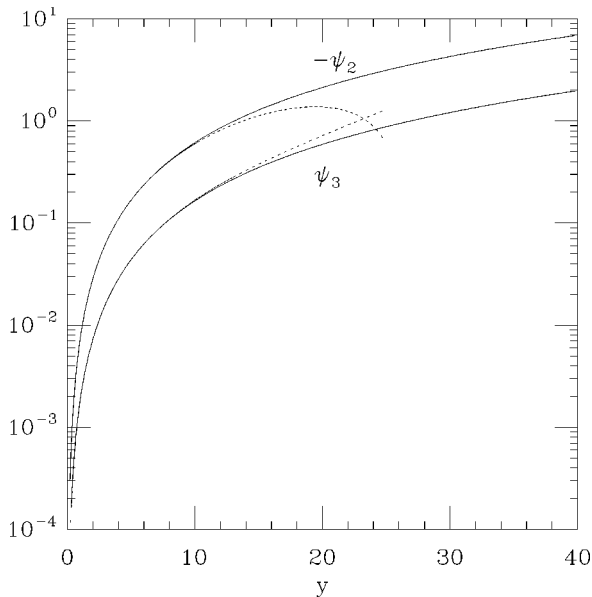


FIG. 1. Plot of the functions $\psi_2(y)$ and $\psi_3(y)$ (full lines) and of their low-momentum expansions up to $O(y^5)$ (dashed lines).

$\psi_2(-1) = -7.72078 \times 10^{-3}$ and $\psi_2'(-1) = 1.56512 \times 10^{-2}$,
 $\psi_3(-1) = 1.89984 \times 10^{-3}$ and $\psi_3'(-1) = -3.8246 \times 10^{-3}$.
 In Fig. 1 we plot the functions $\psi_2(y)$ and $\psi_3(y)$ together with their expansions (73) and (74). Notice the good agreement between the functions $\psi_i(y)$ and their expansions up to $y \approx 9$, which is the theoretical convergence radius of the low-momentum series.

In order to get quantitative estimates from the perturbative ϵ expansion, one should first resum the series and then evaluate the resulting expression at $\epsilon=1$. Usually resummations are performed assuming the Borel summability of the ϵ series. A considerable improvement is obtained if one uses the knowledge of the singularity of the Borel transform [21], $b_s = -(N+8)/3$. As in the analysis of the g expansion, we have used the resummation procedure described in Ref. [36]. Since the ϵ series of c_i , $S_M - 1$ and $S_Z - 1$, have the form $R(\epsilon) = \epsilon^2 \sum_{k=0}^{\infty} a_k \epsilon^k$, we applied the resummation method to $R(\epsilon)$ and to $R(\epsilon)/\epsilon^2$. In Table III we present results for both choices. Since we use a series with only two terms the estimates are not very precise as the large difference between the results obtained with the two methods indicates.

One can also try to get estimates for two-dimensional $O(N)$ models, i.e., for $\epsilon=2$. By resumming the series of $c_2(\epsilon)$ and $S_M(\epsilon)$, we find $c_2 = -0.0010$ and $S_M = 0.9989$ for $N=1$, which must be compared with the exact results [39] $c_2 = -0.000793$ and $S_M = 0.999196$; $c_2 = -0.0013$ and $S_M = 0.9987$ for $N=3$, to be compared with the strong-coupling results [40] $c_2 = -0.0012(2)$ and $S_M = 0.9987(2)$. In both cases the agreement is satisfactory. Instead, when resumming the series divided by ϵ^2 the agreement is poorer. We find $c_2 = -0.0026$ and $S_M = 0.9973$ for $N=1$ and $c_2 = -0.0028$ and $S_M = 0.9971$ for $N=3$. *A posteriori*, it thus appears that the estimates obtained from the resummation of the complete series $R(\epsilon)$ are more reliable. This is confirmed by the three-dimensional analysis where the estimates obtained by considering $R(\epsilon)$ are those that are in better agreement with the strong-coupling and g -expansion estimates.

For quantities that are exactly known in two dimensions,

one can modify the ϵ series to obtain a new expansion, which gives the exact value for $\epsilon=2$. In the case of the two-dimensional Ising model, i.e., $N=1$, the coefficients c_i and S_M have been calculated exactly [39]. So for $N=1$ we can improve the resummation of the ϵ expansion by imposing the exact result for $\epsilon=2$ [41]. One writes

$$R(\epsilon) = R(\epsilon=2) + (2-\epsilon)\bar{R}(\epsilon), \quad (75)$$

and resums the ϵ expansion of $\bar{R}(\epsilon)$. In other words we use as a zeroth-order approximation the linear interpolation of c_i between $d=2$ and $d=4$, and then we use the ϵ series to determine the deviations from the interpolation. As before, one can also apply the same procedure to $R(\epsilon)/\epsilon^2$. We report the results obtained with both choices in Table III. They are referred to as the ‘‘improved’’ ϵ expansion. The estimates are in good agreement with the other results. Notice also that the large discrepancy between the two different resummations of the unconstrained ϵ expansion is here significantly reduced.

E. Strong-coupling analysis

In this subsection we evaluate some of the quantities introduced in Sec. III A by analyzing the strong-coupling expansion of the two-point function $G(x)$ in the lattice N -vector model with nearest-neighbor interactions.

By employing a characterlike approach [42], we have calculated the strong-coupling expansion of $G(x)$ up to 15th order on the cubic lattice and 21st order on the diamond lattice for the corresponding nearest-neighbor formulations. In Appendix B we present the 15th-order strong-coupling expansion of $G(x)$ on the cubic lattice for some values of N . In Appendix C we report the 21st-order strong-coupling series of the magnetic susceptibility and of the second moment of $G(x)$ on the diamond lattice for $N=1,2,3$.

We mention that longer strong-coupling series, up to 21st order, of the lowest moments of $G(x)$ on the cubic and bcc lattices have been recently calculated by a linked-cluster expansion technique, and an updated analysis of the critical exponents γ and ν has been presented [43]. For $N=0$ even longer series have been calculated for various lattices [44–46].

In our strong-coupling analysis, we took special care in devising improved estimators for the physical quantities c_i and S_M , because better estimators can greatly improve the stability of the extrapolation to the critical point. Our search for optimal estimators was guided by the large- N limit of lattice $O(N)$ σ models.

In the large- N limit of the N -vector model on the cubic lattice the following exact relations hold in the high-temperature phase, i.e., for $\beta < \beta_c$,

$$\hat{u}_2^\infty(M_G) \equiv \hat{u}_2 = -\frac{1}{20} M_G^2, \quad (76)$$

$$\hat{u}_3^\infty(M_G) \equiv \hat{u}_3 = \frac{1}{840} M_G^4,$$

etc. \hat{u}_i^∞ vanishes for $T \rightarrow T_c$, i.e., for $M_G^2 \rightarrow 0$, leading to the expected result $c_i = 0$. Similarly on the diamond lattice one obtains

$$\hat{u}_2^\infty(M_G) \equiv \hat{u}_2 = -\frac{1}{20}M_G^2,$$

$$\hat{u}_3^\infty(M_G) \equiv \hat{u}_3 = \frac{1}{7560}M_G^4 \frac{1 + \frac{3}{4}M_G^2}{1 + \frac{1}{12}M_G^2}, \quad (77)$$

etc.

We introduce the quantities

$$\bar{u}_i \equiv \hat{u}_i - \hat{u}_i^\infty(M_G), \quad (78)$$

whose limits for $T \rightarrow T_c$ are still c_i . At $N = \infty$ \bar{u}_i are optimal estimators of c_i , indeed $\bar{u}_i(\beta) = \bar{u}_i^* = c_i = 0$ for $\beta < \beta_c$, i.e., off-critical corrections are absent. It turns out that the use of \bar{u}_i , besides improving the estimates for large values of N , leads also to more precise estimates of c_i at low values of N . Strong-coupling series of \bar{u}_i can be easily obtained from the strong-coupling expansion of $G(x)$. We note that for all values of N and on the cubic lattice, while the series of \hat{u}_2 (\hat{u}_3) starts from β^{-1} (β^{-2}), the series of \bar{u}_2 (\bar{u}_3) starts from β^4 (β^6). A similar fact occurs also on the diamond lattice.

On the lattice, in the absence of a strict rotational invariance, one may actually define different estimators of the mass gap having the same critical limit. On the cubic lattice one may consider μ obtained by the long-distance behavior of the side wall-wall correlation constructed with $G(x)$, or equivalently the solution of the equation $\bar{G}^{-1}(i\mu, 0, 0) = 0$. In view of a strong-coupling analysis, it is convenient to use another estimator of the mass-gap derived from μ [48,6]:

$$M_d^2 = 2(\cosh \mu - 1), \quad (79)$$

which has an ordinary strong-coupling expansion (μ has a singular strong-coupling expansion, starting with $-\ln \beta$). One can easily check that $M_c/\mu \rightarrow 1$ in the critical limit. A similar quantity M_d^2 can be defined on the diamond lattice, as shown in Appendix C [cf. Eq. (C2)].

In order to determine the coefficients c_2 and c_3 of the low-momentum expansion of $\hat{g}_0(y)$ and the mass ratio S_M , we analyzed the strong-coupling series of \bar{u}_2 and \bar{u}_3 [defined in Eq. (78)], and those of the ratios M_c^2/M_G^2 and M_d^2/M_G^2 respectively on the cubic and diamond lattice [47].

In the analysis of a series of the form $A = \beta^m \sum_{i=0}^n a_i \beta^i$, we constructed approximants to the n th-order series $\beta^{-m}A = \sum_{i=0}^n a_i \beta^i$, and then derived the original quantity from them. We considered various types of approximants such as Padé (PA's), Dlog-Padé (DPA's), and first-order inhomogeneous integral approximants (IA's) [49]. In all cases we considered only quasideagonal approximants. We then evaluated them at the critical point β_c in order to obtain an estimate of the corresponding fixed-point value. For the cubic lattice and most values of N , β_c is available in the literature from strong-coupling and numerical Monte Carlo studies (see, for example, Refs. [26,43,45,50–53]). When β_c was not known

(as in the case of diamond lattice models for $N > 0$), we estimated it by performing an IA analysis of the strong-coupling series of the magnetic susceptibility. In our analysis errors due to the uncertainty on the value of β_c turned out to be negligible. The values of β_c used in our calculations are reported in Table II.

In Table III we report our results. The reported estimates of c_2 , c_3 , and S_M summarize the results from all the analyses we performed, and the reported errors are a rough estimate of the uncertainty. The final results are rather accurate taking into account the smallness of the effect we are looking at. Universality among the cubic and diamond lattices is in all cases well verified and gives further support to our final estimates. Our results are in good agreement with the estimates obtained from the other techniques. Only at $N = 0$ are there small discrepancies.

Our strong-coupling analysis represents a substantial improvement with respect to earlier results reported in Ref. [6] for the Ising model, and obtained by an analysis of the strong-coupling series calculated in Refs. [1,55]: $c_2 = -5.5(1.5) \times 10^{-4}$, $c_3 = 0.05(2) \times 10^{-4}$ on the cubic lattice, and $c_2 = -7.1(1.5) \times 10^{-4}$ and $c_3 = 0.09(3) \times 10^{-4}$ on the bcc lattice. Other strong-coupling results can be found in Ref. [48]. Our analysis achieves a considerable improvement with respect to such earlier works essentially for two reasons: we use longer series and improved estimators, see Eq. (78), which allow a more stable extrapolation to the critical limit. Estimates from the analysis of the strong-coupling series of the standard variables \hat{u}_i , defined in Eq. (50), are much less precise, although consistent with those obtained from \bar{u}_i .

F. Conclusions

We have studied the low-momentum behavior of the two-point function in the critical limit by considering several approaches: $1/N$ expansion, g expansion, ϵ expansion and strong-coupling expansion. A summary of our results can be found in Table III.

From the analysis of our strong-coupling series we have obtained quite accurate estimates of the coefficients c_2 , c_3 of the low-momentum expansion (47). Asymptotic large- N formulas (58) and (60) are clearly approached by our strong-coupling results, but only at rather large values of N . The same behavior was already observed for other quantities such as critical exponents [2] and the zero-momentum renormalized four-point coupling [26]. We have also computed the universal function $\hat{g}_0(y)$ in the g expansion in fixed dimension to order $O(g^4)$ and in the ϵ expansion to order $O(\epsilon^3)$. The corresponding estimates of c_2 , c_3 , and S_M are in reasonable agreement with the strong-coupling results.

For all values of N the coefficients c_2 and c_3 turn out to be very small and the pattern (59) is verified. Furthermore the relation (61) is satisfied within the precision of our analysis. A few terms of the expansion of the two-point scaling function $\hat{g}_0(y)$ in powers of y appear to be a good approximation in a relatively large region around $y = 0$, larger than $|y| \lesssim 1$. This is consistent with the fact that the Fourier transform of the two-point function has a simple pole at $k^2 = -M^2$, thus leading to an analytic zero in $\hat{g}_0(y)$ at $y_0 =$

$-S_M$. The pattern of the coefficients c_i suggests that the singularity of $\hat{g}_0(y)$ closest to the origin is much further, which is not unexpected. Indeed we expect that the first singularity of $\hat{g}_0(y)$ is the three-particle cut. In two dimensions, from the exact S matrix [56] one knows that no bound states exist, so that $y_{\text{cut}} = -9S_M$. This is also confirmed by the exactly known two-point function of the $d=2$ Ising model [12]. The $1/N$ expansion of the coefficients c_i suggests that $y_{\text{cut}} \simeq -9$ even in three dimensions.

The few existing Monte Carlo results for the low-momentum behavior of the two-point Green's function are consistent with our determinations but are by far less precise. Using Refs. [57–59] one estimates $c_2 = -13(17) \times 10^{-4}$ for self-avoiding walks, which correspond to $N=0$. In Ref. [60] the authors give a bound on $\sqrt{S_M}$ for the Ising model ($N=1$), from which $-1.2 \times 10^{-3} < S_M - 1 < 0$, which must be compared with our estimate $S_M - 1 = -2.5(5) \times 10^{-4}$. Monte Carlo simulations of the XY model ($N=2$) show that $S_M \simeq 1$ within 0.1% [52], which is consistent with our strong-coupling result $S_M - 1 = -3.5(5) \times 10^{-4}$.

We can conclude that in the critical region of the symmetric phase the two-point Green's function for all N from zero to infinity is almost Gaussian in a large region around $k^2 = 0$, i.e., $|k^2/M_G^2| \lesssim 1$. The small corrections to Gaussian behavior are dominated by the $(k^2)^2$ term in the expansion of the inverse propagator. Via the relation (1) such low-momentum behavior could be probed by scattering experiments by observing the low-angle variation of intensity. A similar low-momentum behavior of the two-point correlation function has been found in two-dimensional $O(N)$ models [39,40,61]. Substantial differences from Gaussian behavior appear at sufficiently large momenta, where $\tilde{G}(k)$ behaves as $1/k^{2-\eta}$ with $\eta \neq 0$ (although η is rather small: $\eta \simeq 0.03$ for $0 \leq N \leq 3$).

The behavior of the two-point function presents a dramatic change in the broken phase. For $N \geq 2$ the transverse and longitudinal magnetic susceptibilities, i.e., the transverse and longitudinal two-point functions at zero momentum, are diverging due to the presence of massless Goldstone bosons. Thus the simple low-momentum expansion found in the symmetric phase does not hold anymore. Only for the Ising model, i.e., for $N=1$, is there a mass gap M in the broken phase. In this case the low-momentum expansion of the scaling two-point function can still be written in the same form as in the symmetric phase. However, now the deviation from a Gaussian behavior is much larger. The coefficients c_i should be larger by about two orders of magnitude [63]. Moreover, by analyzing the low-temperature series published in Ref. [64] one gets $S_M = 0.94(1)$, which compared with the value of $S_M \simeq 0.9997$ obtained in the symmetric phase shows a much larger difference from the Gaussian value $S_M = 1$. The change is even more relevant in the $d=2$ Ising model. Indeed in its broken phase one finds $S_M = 0.3996$, $c_2 = -0.4299$, $c_3 = 0.5256$, etc., which should be compared with the corresponding values in the symmetric phase [39] $S_M = 0.999196$, $c_2 = -7.936 \times 10^{-4}$, $c_3 = 1.095 \times 10^{-5}$, $c_4 = -3.127 \times 10^{-7}$, etc. Moreover the singularity at $k^2 = -M^2$ of $\tilde{G}(k)$ is not a simple pole, but a cut. As a consequence the corresponding zero in $\hat{g}_0(y)$ is not analytic, and therefore the

convergence radius of the expansion around $y=0$ should be S_M .

IV. ANISOTROPY OF $G(x)$ AT LOW MOMENTUM AND IN THE CRITICAL REGIME

In this section we will study anisotropic effects on the two-point function due to the lattice structure. We will mainly consider three-dimensional lattices with cubic symmetry. However, whenever possible, we will give expressions for general d -dimensional lattices with hypercubic symmetry, so that one can recover the results for the square lattice and compare with perturbative series in $d=4-\epsilon$. We will also comment briefly and present some results for the triangular, honeycomb, and diamond lattices.

A. Notations

In the following subsections we will compute the exponent $\rho = 2 + \eta - \eta_4$ defined in Eq. (29). It can be derived directly from Eq. (26) or Eq. (28) or by studying the weighted moments $\bar{q}_{4,j} = q_{4,j}/m_0$ where $q_{4,j}$ is defined in Eq. (33) and $m_0 \equiv \chi$. Indeed for $M_G \rightarrow 0$,

$$\bar{q}_{4,j} \sim M_G^{-4-2j+\rho}. \quad (80)$$

We will also compute the universal function $\hat{g}_4(y)$. In particular we will be interested in the first terms of its expansion in powers of y around $y=0$:

$$\hat{g}_4(y) = 1 + \sum_{i=1} d_i y^i, \quad (81)$$

where $d_i \equiv c_{4,i}$ [cf. Eq. (25)]. The coefficients d_i can be easily obtained from the expressions of the moments $q_{4,m}$. For $M_G \rightarrow 0$, we find

$$\frac{\bar{q}_{4,1}}{q_{4,0}} \rightarrow 4(d+8)(1 - \frac{1}{2}d_1)M_G^{-2},$$

$$\frac{\bar{q}_{4,2}}{q_{4,0}} \rightarrow 24(d+8)(d+10)(1 - \frac{2}{3}d_1 - \frac{2}{3}c_2 + \frac{1}{3}d_2)M_G^{-4}, \quad (82)$$

and so on. From Eq. (82) it is easy to derive expressions for $r_i \equiv u_{4,i}/u_{4,0}$ whose critical limit is d_i . In particular

$$r_1 = 2 - \frac{M_G^2}{2(d+8)} \frac{\bar{q}_{4,1}}{q_{4,0}}. \quad (83)$$

B. Breaking of rotational invariance in the large- N limit

In the large- N limit lattice $O(N)$ models become massive Gaussian theories that can be solved exactly. If one considers theories defined on Bravais lattices one has in the large- N limit

TABLE IV. Three-dimensional N -vector model with nearest-neighbor interactions: lowest moments of $G(x)$ at $N=\infty$ on the cubic, fcc, and diamond lattice. z is the inverse of the second moment correlation length.

Moments	Cubic	fcc	Diamond
χ	$\frac{1}{\beta z}$	$\frac{1}{2\beta z}$	$\frac{3}{2\beta z}$
\bar{m}_2	$\frac{6}{z}$	$\frac{6}{z}$	$\frac{6}{z}$
M_G^2	z	z	z
$\bar{q}_{3,0}$	0	0	$\frac{1}{6\sqrt{3}}\left(1+\frac{z}{12}\right)^{-1}$
\bar{m}_4	$\frac{120}{z^2}\left(1+\frac{z}{20}\right)$	$\frac{120}{z^2}\left(1+\frac{z}{20}\right)$	$\frac{120}{z^2}\left(1+\frac{z}{20}\right)$
$\bar{q}_{4,0}$	$\frac{12}{5z}$	$-\frac{3}{5z}$	$-\frac{8}{5z}$
$\bar{q}_{3,1}$	0	0	$\sqrt{3}\left(1+\frac{z}{18}+\frac{z^2}{216}\right)\left(1+\frac{z}{12}\right)^{-2}$
\bar{m}_6	$\frac{5040}{z^3}\left(1+\frac{z}{10}+\frac{z^2}{840}\right)$	$\frac{5040}{z^3}\left(1+\frac{z}{10}+\frac{z^2}{840}\right)$	$\frac{5040}{z^3}\left(1+\frac{11z}{60}+\frac{8z^2}{945}+\frac{z^3}{10080}\right)$
$\bar{q}_{4,1}$	$\frac{528}{5z^2}\left(1+\frac{z}{44}\right)$	$-\frac{132}{5z^2}\left(1+\frac{z}{44}\right)$	$-\frac{352}{5z^2}\left(1+\frac{z}{33}+\frac{z^2}{528}\right)\left(1+\frac{z}{12}\right)^{-1}$
$\bar{q}_{6,0}$	$\frac{12}{77z}$	$-\frac{39}{154z}$	$-\frac{416}{231z}\left(1-\frac{z}{78}\right)\left(1+\frac{z}{12}\right)^{-1}$
\bar{m}_8	$\frac{362880}{z^4}\left(1+\frac{3z}{20}+\frac{11z^2}{2160}+\frac{z^3}{60480}\right)$	$\frac{362880}{z^4}\left(1+\frac{3z}{20}+\frac{37z^2}{7560}+\frac{z^3}{60480}\right)$	$\frac{362880}{z^4}\left(1+\frac{3z}{20}+\frac{389z^2}{136080}+O(z^3)\right)$
$\bar{q}_{4,2}$	$\frac{41184}{5z^3}\left(1+\frac{19z}{286}+\frac{z^2}{3432}\right)$	$-\frac{10296}{5z^3}\left(1+\frac{9z}{143}+\frac{z^2}{3432}\right)$	$-\frac{27456}{5z^3}\left(1+\frac{z}{858}+\frac{139z^2}{30888}+O(z^3)\right)$
$\bar{q}_{6,1}$	$\frac{240}{11z^2}\left(1+\frac{z}{140}\right)$	$-\frac{1110}{77z^2}\left(1+\frac{13z}{740}\right)$	$-\frac{23680}{231z^2}\left(1-\frac{129z}{1480}+\frac{5z^2}{444}+O(z^3)\right)$

$$\tilde{G}^{-1}(k) = c\beta(\bar{k}^2 + M_G^2), \quad (84)$$

where \bar{k}^2 is defined by Eq. (8). The relation between M_G^2 and β is given by the gap equation. The constant c is lattice dependent and will not play any role in the discussion. The function \bar{k}^2 has the properties mentioned at the beginning of Sec. II A and a multipole expansion of the type (10) for lattices with cubic symmetry. For other Bravais lattices the only difference is the presence of different multipole combinations. Considering first lattices with (hyper) cubic symmetry, from Eqs. (10) and (14), we find for $M_G \rightarrow 0$

$$\tilde{G}^{-1}(k) = c\beta M_G^2 \{1 + y + M_G^2 [e_{2,0}y^2 + e_{4,0}Q_4(k/M_G)] + \dots\}. \quad (85)$$

Comparing with Eqs. (20) and (28) we get immediately $\rho = 2$ and $\hat{g}_4(y) = 1$, i.e., $d_i = 0$ for all $i \neq 0$.

In the large- N limit one can easily verify the universality properties of the ratios defined in Eq. (34). Indeed for generic Hamiltonians in the critical limit $M_G \rightarrow 0$ (keeping the dimension of the lattice d generic) we have

$$\bar{m}_{2m} \rightarrow 2^m m! \left(\prod_{i=0}^{m-1} (d+2i) \right) M_G^{-2m}, \quad (86)$$

$$\frac{\bar{q}_{4,m}}{\bar{q}_{4,0}} \rightarrow 2^m (m+1)! \left(\prod_{i=0}^{m-1} (d+8+2i) \right) M_G^{-2m}, \quad (87)$$

and

$$\bar{q}_{4,0} \rightarrow -e_{4,0} \frac{24d(d-1)}{d+2} M_G^{-2}. \quad (88)$$

Notice that the only dependence on the specific Hamiltonian is in the expression of $\bar{q}_{4,0}$. (Exact expressions for some of these quantities are reported for the theory with nearest-neighbor interactions on the cubic, diamond, and fcc lattice in Table IV and on the square lattice in Table V.) Universality is then a straightforward consequence of the independence of the ratio (87) from $e_{4,0}$. It should also be noticed that $\bar{q}_{4,m}/\bar{m}_{4+2m} \sim M_G^2$. This shows that, as expected, anisotropic moments are suppressed by two powers of M_G in

TABLE V. Two-dimensional N -vector model with nearest-neighbor interactions: lowest moments of $G(x)$ at $N=\infty$ on the square, triangular, and honeycomb lattice. $z=M_G^2$.

Moments	Square	Triangular	Honeycomb
χ	$\frac{1}{\beta z}$	$\frac{2}{3\beta z}$	$\frac{4}{3\beta z}$
\bar{m}_2	$\frac{4}{z}$	$\frac{4}{z}$	$\frac{4}{z}$
M_G^2	z	z	z
$\bar{t}_{3,0}$	0	0	$\frac{1}{2}\left(1+\frac{z}{8}\right)^{-1}$
\bar{m}_4	$\frac{64}{z^2}\left(1+\frac{z}{16}\right)$	$\frac{64}{z^2}\left(1+\frac{z}{16}\right)$	$\frac{64}{z^2}\left(1+\frac{z}{16}\right)$
$\bar{q}_{4,0}$	$\frac{1}{z}$	0	0
\bar{m}_6	$\frac{2304}{z^3}\left(1+\frac{z}{8}+\frac{z^2}{576}\right)$	$\frac{2304}{z^3}\left(1+\frac{z}{8}+\frac{z^2}{576}\right)$	$\frac{2304}{z^3}\left(1+\frac{z}{4}+\frac{z^2}{64}+\frac{z^3}{4608}\right)\left(1+\frac{z}{8}\right)^{-1}$
$\bar{q}_{4,1}$	$\frac{40}{z^2}\left(1+\frac{z}{40}\right)$	0	0
$\bar{t}_{6,0}$	0	$-\frac{4}{z}$	$\frac{36}{z}\left(1-\frac{z}{72}\right)\left(1+\frac{z}{8}\right)^{-1}$

agreement with the prediction $\rho=2$. We stress that the universality of $R_{4,m,n}$ is due the fact that there is only one leading irrelevant operator breaking rotational invariance.

It is interesting to notice that such a universality does not hold for moments $\bar{q}_{6,m}$ (or for $\bar{q}_{2l,m}$ for higher values of l) because of the mixings we have mentioned in Sec. II A. For $\bar{q}_{6,m}$ we have for $T\rightarrow T_c$

$$\frac{\bar{q}_{6,m}}{\bar{q}_{6,0}} \rightarrow 2^m(m+1)! \left(1 + \frac{e_{4,0}^2}{e_{6,0}} \frac{8m}{d+12}\right) \times \left(\prod_{i=0}^{m-1} (d+12+2i)\right) M_G^{-2m}, \quad (89)$$

which depends on $e_{6,0}$ and $e_{4,0}^2$, a consequence of the fact that $Q_4(k)^2$ contains a term of the form $k^2 Q_6(k)$. Thus ratios of the form (34) built with $\bar{q}_{6,m}$ are not universal.

Let us now consider the diamond lattice. In this case not only is rotational invariance broken, but also parity symmetry. As the leading anisotropic operator is $O_{4,0}(x)$ the behavior of the leading anisotropic corrections is identical to that we have discussed above. Therefore $\rho=2$ also in this case. Moreover the invariant ratios $R_{4,m,n}$ are identical for the diamond lattice and for the other Bravais lattices with cubic symmetry. Equation (87) is exact for the diamond lattice as well.

To discuss parity-breaking effects we must consider odd moments of $G(x)$. In particular one finds that, for $M_G\rightarrow 0$,

$$\bar{q}_{3,0} \equiv \frac{q_{3,0}}{m_0} \rightarrow \frac{1}{6\sqrt{3}}, \quad (90)$$

where $q_{3,0} \equiv \sum xyz G(x,y,z)$. Thus parity-breaking effects vanish as M_G^3 , i.e., $\rho_p=3$, faster than the anisotropic effects we have considered previously.

Finally let us consider lattices that do not have cubic invariance, such as the triangular and the honeycomb one. In Table V we report the large- N limit of some of the lowest spherical and nonspherical moments of $G(x)$ for the models with nearest-neighbor interactions.

For the triangular lattice one should consider the multipole expansion (36). In this case the leading term breaking rotational invariance is proportional to $T_6(k)$ and thus we have $\rho=4$. This is indeed confirmed by the fact that, for $M_G\rightarrow 0$, $\bar{t}_{6,m}/\bar{m}_{6+2m} \sim M_G^4$, where $\bar{t}_{6,m} = t_{6,m}/m_0$ and $t_{6,m}$ is defined in Eq. (37). As in the cubic case, it is immediate to verify the universality of ratios of the form given in Eq. (34) with $t_{6,m}$ instead of $q_{4,m}$, which is a consequence of the uniqueness of the leading operator breaking rotational invariance. Universality follows from the fact that, for $T\rightarrow T_c$,

$$\frac{\bar{t}_{6,m}}{\bar{t}_{6,0}} \rightarrow \frac{2^{2m}(m+1)!(m+5)!}{5!} M_G^{-2m}, \quad (91)$$

independently of the specific Hamiltonian.

For the honeycomb lattice one must also consider the breaking of parity. Considering the odd moment $t_{3,0} = \sum(x^3 - 3y^2x)G(x,y)$ [cf. Eq. (42)], one finds $\bar{t}_{3,0} \equiv t_{3,0}/m_0 \rightarrow \frac{1}{2}$. Thus, as in the diamond case, parity breaking effects vanish as M_G^3 , i.e., $\rho_p=3$.

C. Analysis to order $1/N$

In the previous subsection we computed the exponent ρ for $N\rightarrow\infty$ for lattices with cubic symmetry, finding $\rho=2$.

Now we want to compute the $1/N$ corrections, i.e., the value of $\sigma \equiv \sigma_4 = \eta - \eta_4$ [cf. Eq. (29)], which is the anomalous dimension of the operator $O_{4,0}(x)$. More generally we can compute the exponents η_{2l} defined in Eq. (26) for arbitrary l . Notice that in this way we will also obtain the $1/N$ correction to ρ for the triangular lattice that depends on η_6 .

In d dimensions, we consider the following representation of the inverse two-point function where the $O(1/N)$ correction has been included:

$$\beta^{-1}\tilde{G}^{-1}(k) = \beta^{-1}Z_G^{-1}M_G^2 + \bar{k}^2 + \frac{1}{N} \int \frac{d^d p}{(2\pi)^d} \bar{\Delta}(p) \times \left(\frac{1}{p+k^2+M_G^2} - \frac{1}{p^2+M_G^2} \right). \quad (92)$$

Here \bar{k}^2 is the inverse lattice propagator defined in Eq. (8), Z_G is the constant defined in Eq. (52), and

$$\bar{\Delta}^{-1}(p) = \frac{1}{2} \int \frac{d^d q}{(2\pi)^d} \frac{1}{(q+p^2+M_G^2)(q^2+M_G^2)}. \quad (93)$$

The following statements can be checked explicitly in Eq. (92) and hold to all orders of the $1/N$ expansion: (i) in the limit $M_G \rightarrow 0$ the function $\tilde{G}^{-1}(k, M_G)$ is spherically symmetric (i.e., it depends only on $y \equiv k^2/M_G^2$, apart from an overall factor); (ii) the only nonspherically symmetric contribution that may appear in $\tilde{G}^{-1}(k, M_G)$ to $O(M_G^4)$ can be reduced to a spherically symmetric function multiplied by $Q_4(k)$. These statements are simply a consequence of applying the discrete and continuous symmetry properties to all integrals appearing in the asymptotic expansion in M_G of the relevant Feynman integrals. They prove to all orders in $1/N$ the validity of the expansion (28).

To compute the anomalous dimension η_{2l} to order $1/N$ we will use the trick we explained in Sec. II A. If one considers a particular Hamiltonian such that $g_{2l}(y, M_G) = 0$ for $0 \leq l \leq \bar{l}$, then $\tilde{G}^{-1}(k)$ has an expansion of the form (28) with $\eta_4 \rightarrow \eta_{2\bar{l}}$ and $\hat{g}_4(y) \rightarrow \hat{g}_{2\bar{l}}(y)$. In the $1/N$ expansion, to order $1/N$ this can be achieved by considering Hamiltonians such that, for $k \rightarrow 0$ (to simplify the notation from now on we write l instead of \bar{l}),

$$\bar{k}^2 = k^2 + rk^{2l} + O(k^{2l+2}), \quad (94)$$

where $k^{2l} \equiv \sum_{\mu} k_{\mu}^{2l}$. The limit $M_G \rightarrow 0$ can then be easily obtained by evaluating massless continuum integrals, and taking the contribution proportional to r , which is the only term relevant to our computation. In this limit we obtain

$$\bar{\Delta}^{-1}(p) \approx \Delta_0^{-1}(p) \left(1 - rB_l \frac{p^{2l}}{p^2} \right), \quad (95)$$

where

$$\Delta_0^{-1}(p) = \frac{1}{2} (p^2)^{d/2-2} \frac{\Gamma(2-d/2)\Gamma(d/2-1)^2}{(4\pi)^{d/2}\Gamma(d-2)}, \quad (96)$$

$$B_l = (4-d) \frac{\Gamma(d/2+2l-2)\Gamma(d-2)}{\Gamma(d+2l-3)\Gamma(d/2-1)}, \quad (97)$$

and we have discarded rotationally invariant terms proportional to r , since they will not contribute to the final result.

We must now identify the singular contribution in the limiting form of Eq. (92):

$$\begin{aligned} \beta^{-1}\tilde{G}^{-1}(k) &\rightarrow k^2 + rk^{2l} + \frac{1}{N} \int \frac{d^d p}{(2\pi)^d} \Delta_0(p) \left[1 - rB_l \frac{p^{2l}}{p^2} \right]^{-1} \\ &\quad \times [(p+k)^2 + r(p+k)^{2l}]^{-1} \\ &\approx k^2 \left(1 - \frac{1}{N} \eta_l \ln k \right) + rk^{2l} \left(1 - \frac{1}{N} \eta_{2l,1} \ln k \right). \end{aligned} \quad (98)$$

The coefficients η_1 and $\eta_{2l,1}$ are related to the $1/N$ expansion of the exponents η and η_{2l} :

$$\eta = \frac{\eta_1}{N} + O\left(\frac{1}{N^2}\right), \quad (99)$$

$$\eta_{2l} = \frac{\eta_{2l,1}}{N} + O\left(\frac{1}{N^2}\right). \quad (100)$$

By simple manipulations one obtains

$$\eta_1 = - \frac{4\Gamma(d-2)}{\Gamma(2-d/2)\Gamma(d/2-2)\Gamma(d/2-1)\Gamma(d/2+1)}, \quad (101)$$

and

$$\eta_{2l,1} = \frac{d(d-2)}{(d-2+4l)(d-4+4l)} \left[1 + 2 \frac{\Gamma(2l+1)\Gamma(d-2)}{\Gamma(2l+d-3)} \right] \eta_1. \quad (102)$$

Therefore for $d=3$ we find

$$\sigma = \eta - \eta_4 = \frac{32}{21\pi^2 N} + O\left(\frac{1}{N^2}\right). \quad (103)$$

Note that the coefficient of the leading $1/N$ term is very small. Thus, at least for N sufficiently large, say $N \geq 8$, where the $1/N$ expansion is known to work reasonably well, corrections to the Gaussian value of ρ are very small.

For $d \rightarrow 2$, $\eta_{2l,1} \rightarrow \eta_1$. Therefore in two dimensions and to $O(1/N)$, there are no corrections to the Gaussian value, i.e., the first coefficient of the expansion of the anomalous dimension is zero to $O(1/N)$. One might only observe (suppressed) logarithmic corrections to canonical scaling for all l . It is easy to check in perturbation theory that this holds exactly for all $N \geq 3$.

The computation of the universal function $\hat{g}_4(y)$ is particularly involved. The result is given in Appendix A. Here we will only give the values of the coefficients d_i of its low-momentum expansion [cf. Eq. (81)]. We found

$$d_i = \frac{\bar{d}_i}{N} + O\left(\frac{1}{N^2}\right), \quad (104)$$

where $\bar{d}_1 = -0.002\,064\,68$, $\bar{d}_2 = 0.000\,073\,78$, $\bar{d}_3 = -0.000\,004\,24$, etc.

D. g expansion analysis

The critical exponent σ and the scaling function $\hat{g}_4(y)$ can also be evaluated in the g expansion. For this purpose we calculated the one-particle irreducible two-point function $\Gamma_{O_4}(k, M_G)$ defined in Eq. (22). By a three-loop calculation one finds

$$\begin{aligned} \Gamma_{O_4}(k, M_G) &= Q_4(k) + g_0^2 \frac{N+2}{6} J_2(k, M_G) \\ &\quad - g_0^3 \frac{(N+2)(N+8)}{108} [J_{3,1}(k, M_G) \\ &\quad + 4J_{3,2}(k, M_G)] + O(g_0^4), \end{aligned} \quad (105)$$

where

$$J_2(k, m) = \int \frac{d^3 p}{(2\pi)^3} \frac{Q_4(k-p)A(p, m)}{[(k-p)^2 + m^2]^2}, \quad (106)$$

$$J_{3,1}(k, m) = \int \frac{d^3 p}{(2\pi)^3} \frac{Q_4(k-p)A(p, m)^2}{[(k-p)^2 + m^2]^2}, \quad (107)$$

$$J_{3,2}(k, m) = \int \frac{d^3 p}{(2\pi)^3} \frac{A(p, m)A_Q(p, m)}{(k-p)^2 + m^2}, \quad (108)$$

and

$$\begin{aligned} A(p, m) &= \int \frac{d^3 q}{(2\pi)^3} \frac{1}{[q^2 + m^2][(q+p)^2 + m^2]} \\ &= \frac{1}{4\pi p} \arctan \frac{p}{2m}, \end{aligned} \quad (109)$$

$$A_Q(p, m) = \int \frac{d^3 q}{(2\pi)^3} \frac{Q_4(q)}{[q^2 + m^2]^2 [(q+p)^2 + m^2]}. \quad (110)$$

By renormalizing $\Gamma_{O_4}(k, M_G)$ at $k=0$ according to Eqs. (22)–(24), one obtains the corresponding renormalization constant \bar{Z}_4 and renormalized function $\Gamma_{O_4, R}(k, M_G)$. The critical exponent σ is obtained by evaluating the anomalous dimension

$$\begin{aligned} \gamma_{O_4}(\bar{g}) &= \beta(\bar{g}) \frac{\partial \ln(\bar{Z}_4/Z_G)}{\partial \bar{g}} \\ &= \bar{g}^2 \frac{5408}{25515} \frac{N+2}{(N+8)^2} [1 + \bar{g} \times 0.045007 + O(\bar{g}^2)] \end{aligned} \quad (111)$$

at the fixed-point value of the coupling, i.e., $\sigma = \gamma_{O_4}(g^*)$.

TABLE VI. For various values of N , we report estimates of σ obtained by our strong-coupling analysis, from the $1/N$ expansion, from the resummation of the g expansion (see Sec. III D) [in this case we give two numbers corresponding to the two choices: resumming $R(x)$ or $R(x)/x^2$], and from the $O(\epsilon^2)$ term of the ϵ expansion.

N	s.c. expansion	$1/N$ expansion	g expansion	ϵ expansion
0	0.00(1)		0.0119, 0.0141	0.0109
1	0.01(1)		0.0143, 0.0166	0.0130
2	0.02(1)		0.0156, 0.0177	0.0140
3	0.03(2)	0.0515	0.0160, 0.0179	0.0145
4	0.03(2)	0.0386	0.0158, 0.0174	0.0147
8	0.02(1)	0.0193	0.0139, 0.0148	0.0137
16	0.009(3)	0.0096	0.0098, 0.0109	0.0109
32	0.004(2)	0.0048	0.0058, 0.0059	0.0074

The scaling function $\hat{g}_4(y)$ is obtained from the zero-momentum renormalized function $\Gamma_{O_4, R} \equiv Q_4(k)f_4(g, y)$, by $\hat{g}_4(y) = f_4(g^*, y)$. By expanding $f_4(g, y)$ in powers of y around $y=0$, one finds

$$d_i = \bar{g}^2 \frac{N+2}{(N+8)^2} \bar{d}_i \quad (112)$$

and

$$\begin{aligned} \bar{d}_1 &= -\frac{380}{168\,399} [1 + \bar{g} \times 0.105400 + O(\bar{g}^2)], \\ \bar{d}_2 &= \frac{3076}{197\,026\,83} [1 - \bar{g} \times 0.355629 + O(\bar{g}^2)], \end{aligned} \quad (113)$$

$$\bar{d}_3 = -\frac{3112}{253\,320\,210} [1 - \bar{g} \times 0.696450 + O(\bar{g}^2)],$$

etc.

In order to get estimates of σ and of the coefficients d_i from the corresponding series, we have employed the resummation procedure used in Sec. III C. Results for σ are reported in Table VI, and for d_1 in Table III.

E. An ϵ -expansion analysis

To compute the exponents η_{2l} and $\hat{g}_4(y)$ in the framework of the ϵ expansion, we again calculated the renormalized two-point one-particle irreducible function with an insertion of the operator $O_4(x)$; see Eq. (24). To order $O(\epsilon^2)$ we find

$$\sigma = \eta - \eta_4 = \frac{7}{20} \frac{N+2}{(N+8)^2} \epsilon^2 + O(\epsilon^3) \quad (114)$$

and

$$\begin{aligned} \hat{g}_4(y) &= 1 + \epsilon^2 \frac{N+2}{(N+8)^2} 8\pi^4 [Q_4(\partial/\partial k)J_s(k, 1) \\ &\quad - Q_4(\partial/\partial k)J_s(k, 1)|_{k=0}] + O(\epsilon^3). \end{aligned} \quad (115)$$

The function $J_s(k, m)$ is the finite part of the integral

$$J(k, m) = \int \frac{d^d q}{(2\pi)^d} \frac{d^d p}{(2\pi)^d} \times \frac{Q_4(k-p)}{[q^2 + m^2][(q+p)^2 + m^2][(k-p)^2 + m^2]^2} \quad (116)$$

with the modified minimal subtraction ($\overline{\text{MS}}$) prescription. The expansion of $\hat{g}_4(y)$ in powers of y gives

$$d_i = \epsilon^2 \frac{N+2}{(N+8)^2} \hat{d}_i + O(\epsilon^3) \quad (117)$$

and $\hat{d}_1 = -0.003\,545\,00$, $\hat{d}_2 = 0.000\,117\,15$, $\hat{d}_3 = -0.000\,005\,99$, etc.

F. A strong-coupling analysis

Anisotropy in the two-point function can be studied for finite values of N by analyzing the strong-coupling expansion of its lowest nonspherical moments.

In order to compute σ , the correction to the Gaussian value of ρ , we analyze the strong-coupling expansion of the ratio $q_{4,0}/m_2$, which behaves as

$$\frac{q_{4,0}}{m_2} \sim M_G^\sigma \sim (T - T_c)^{\sigma\nu} \quad (118)$$

for $T \rightarrow T_c$. We recall that in the $1/N$ expansion $\nu = 1 + O(1/N)$, and for $N=0,1,2,3$ $\nu \approx 0.588$, $\nu \approx 0.630$, $\nu \approx 0.670$, $\nu \approx 0.705$, respectively [2]. DPA's and IA's of the available strong-coupling series of the ratio $q_{4,0}/m_2$ on both cubic and diamond lattices turned out not to be sufficiently stable to provide satisfactory estimates of σ at any finite value of N .

A better analysis has been obtained by employing the so-called critical point renormalization method (CPRM) [62]. The idea of the CPRM is the following: start from two series $D(x)$ and $E(x)$, which are singular at the same point x_0 ,

$$D(x) = \sum_i d_i x^i \sim (x_0 - x)^{-\delta}, \quad (119)$$

$$E(x) = \sum_i e_i x^i \sim (x_0 - x)^{-\epsilon},$$

and construct a new series by

$$F(x) = \sum_i \frac{d_i}{e_i} x^i. \quad (120)$$

The function $F(x)$ is singular at $x=1$ and for $x \rightarrow 1$ behaves as $F(x) \sim (1-x)^{-\phi}$, where $\phi = 1 + \delta - \epsilon$. Therefore the analysis of $F(x)$ provides an unbiased estimate of the difference between the critical exponents of the two functions $D(x)$ and $E(x)$. Moreover the series $F(x)$ may be analyzed by employing biased approximants with a singularity at $x_0 = 1$.

By applying the CPRM to the strong-coupling series of $q_{4,0}$ and m_2 , one can extract an unbiased estimate of σ by computing the exponent $\phi = 1 - \sigma\nu$ from the resulting series at the singularity $x_0 = 1$. We analyzed this series by biased IA's. The estimates of σ we obtained confirm universality between the cubic and the diamond lattice, although the analysis on the diamond lattice led in general to less stable results. In Table VI, for selected values of N , we report our estimates of σ , which are essentially obtained from the analysis on the cubic lattice. In order to derive σ from $\sigma\nu$, which is the quantity derived from the strong-coupling analysis, we have used the values of ν available in the literature. See, e.g., Ref. [43] for an updated collection of results obtained by various numerical and analytic methods. The errors we report are rough estimates of the uncertainty obtained by considering the spread of all the analyses we performed. The values of σ are very small for all values of N , and for large N , say $N \geq 10$, they are consistent with the corresponding $O(1/N)$ prediction, cf. Eq. (103).

In order to estimate the first nontrivial coefficient d_1 of the expansion of $\hat{g}_4(y)$, see Eq. (81), one may consider the quantity r_1 defined in Eq. (83). However, as we did for the analysis of c_i in Sec. III, it is better to consider another quantity \bar{r}_1 which is defined so that $\bar{r}_1 = 0$ for $N = \infty$ for all $\beta < \beta_c$. For the cubic lattice

$$\bar{r}_1 = 2 - \frac{q_{4,1} M_G^2}{22q_{4,0}} + \frac{M_G^2}{22}, \quad (121)$$

while for the diamond lattice

$$\bar{r}_1 = 2 \frac{1 + \frac{2}{33} M_G^2 + \frac{1}{528} M_G^4}{1 + \frac{1}{12} M_G^2} - \frac{q_{4,1} M_G^2}{22q_{4,0}}. \quad (122)$$

In the critical limit $\bar{r}_1 \rightarrow d_1$. The estimates of d_1 obtained from the analysis of the strong-coupling series of \bar{r}_1 [65] are reported in Table III. Universality between the cubic and diamond lattice is again substantially verified, although the diamond lattice provides in most cases less precise results. The value of d_1 is very small for all values of N . At large- N the strong-coupling estimate of d_1 is in good agreement with the large- N prediction (104). The estimates are also in satisfactory agreement with the results obtained from the g expansion and the ϵ expansion.

Finally we compute ρ_p for the diamond lattice. For a Gaussian theory $\rho_p = 3$ and thus $q_{3,0} \rightarrow \text{const}$ for $M_G \rightarrow 0$. In general, for finite values of N , we write $\rho_p = 3 + \sigma_p$. The exponent σ_p is determined from the critical behavior of $\bar{q}_{3,0}$, indeed $\bar{q}_{3,0} \sim M_G^{\sigma_p}$. In order to estimate σ_p , we applied the CPRM to the series $q_{3,0}$ and χ . We found $0 \leq \sigma_p \leq 0.01$ for all $N \geq 0$.

G. The two-dimensional Ising model

We conclude this section by considering the two-dimensional Ising model, for which we present an argument showing that the anomalous dimension of the irrelevant operators breaking rotational invariance is zero.

Let us consider first the square lattice. In this case, for sufficiently large values of $|x|$ the asymptotic behavior of $G(x)$ on the square lattice can be written in the form [66]

$$G(x) \approx \int \frac{d^2 p}{(2\pi)^2} e^{i\vec{p}\cdot\vec{x}} \frac{Z(\beta)}{M^2(\beta) + \hat{p}^2}, \quad (123)$$

where $\hat{p}^2 = \sum_{\mu} 4 \sin^2(p_{\mu}/2)$,

$$Z(\beta) = [(1-z^2)^2 - 4z^2]^{1/4} \frac{(1+z^2)^{1/2}}{z}, \quad (124)$$

and

$$M^2(\beta) = \frac{(1+z^2)^2}{z(1-z^2)} - 4, \quad (125)$$

and we have introduced the auxiliary variable $z(\beta) = \tanh\beta$. This shows that at large distances the breaking of rotational invariance is identical to that of the massive Gaussian model with nearest-neighbor interactions. Therefore $\rho=2$ exactly.

This value of ρ is confirmed by a strong-coupling analysis of the moments $q_{4,m}$ using the available 21st-order strong-coupling series [39]. In particular, on the square lattice we found $q_{4,0}/m_2 \rightarrow 1/4$ for $\beta \rightarrow \beta_c$ within an uncertainty of $O(10^{-5})$.

A formula analogous to Eq. (123) has been conjectured in Ref. [39] for the Ising model on triangular and honeycomb lattices. Thus, also on these lattices, the pattern of breaking of rotation invariance (and parity in the case of the honeycomb lattice) should be that of the corresponding Gaussian theories, which have been described in Sec. IV B. If the conjecture of Ref. [39] is correct, we have $\rho=4$ for the triangular lattice and $\rho_p=3$ for the honeycomb lattice.

Again, an analysis of the strong-coupling expansion of $G(x)$ on the triangular and honeycomb lattices supports convincingly this conjecture.

H. Conclusions

For lattice models with $O(N)$ symmetry we studied the problem of the recovery of rotational invariance in the critical limit. Anisotropic effects vanish as M_G^{ρ} , when $M_G \rightarrow 0$. The universal critical exponent ρ , which is related to the critical dimension of the leading operator breaking rotational

invariance, turns out to be 2 with very small N -dependent corrections for the lattices with cubic symmetry. Notice that this behavior is universal and thus should appear in all physical systems that have cubic symmetry. The reader should note that ρ is different from the exponent ω , which parametrizes the leading correction to scaling and which is related to a different rotationally invariant irrelevant operator. Models defined on lattices with basis, such as the diamond lattice, show also a breaking of the parity symmetry. We find that these effects vanish as $M_G^{\rho_p}$, with $\rho_p \approx 3$ for all values of N .

We have also calculated the universal function $\hat{g}_4(y)$. For $y \lesssim 1$, we find $\hat{g}_4(y) \approx 1$ with very small corrections.

In our study we considered several approaches, based on $1/N$, g , ϵ , and strong-coupling expansions. All results are in good agreement.

In two dimensions we showed that $\rho=2$ for the square lattice for all $N \geq 3$ and $N=1$. We conjecture that this is a general result, valid for all values of N . Similar arguments apply to the triangular (honeycomb) lattice: we conjecture $\rho=4$ ($\rho_p=3$) for all N .

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APPENDIX A: $O(1/N)$ CALCULATIONS

In this Appendix we present a simple derivation of all the results that are needed in order to construct explicitly the $1/N$, g , and ϵ expansions up to three loops presented in Sec. III. Our starting point is the observation that most of the two- and three-loop calculations needed in the relevant perturbative calculations are included, apart from rather trivial algebraic dependences on N , in the one-loop calculation of the $1/N$ expansion for the two-point function. As we shall show, the $1/N$ results can be expanded in g and ϵ in order to recover all the corresponding contributions. Let us therefore start with the evaluation of the renormalized self-energy to $O(1/N)$ in arbitrary dimension d and for arbitrary bare coupling g_0 in the N -component ϕ^4 theory.

We introduce the dressed composite propagator (geometric sum of bubble insertions in the ϕ^4 vertex):

$$\Delta^{-1}(y, g_0) \equiv \left[\frac{1}{2} \int \frac{d^d p}{(2\pi)^d} \frac{1}{p^2 + m^2} \frac{1}{(p+k)^2 + m^2} + \frac{3}{Ng_0} \right] m^{4-d} \equiv \Delta_r^{-1}(y) + \frac{3}{Ng}, \quad (A1)$$

where $y \equiv k^2/m^2$, and we have defined the (zero-momentum subtracted) dimensionless renormalized dressed (inverse) propagator:

$$\Delta_r^{-1}(y) \equiv m^{4-d} \frac{1}{2} \int \frac{d^d p}{(2\pi)^d} \frac{1}{p^2 + m^2} \left[\frac{1}{(p+k)^2 + m^2} - \frac{1}{p^2 + m^2} \right], \quad (A2)$$

and the four-point (large- N) coupling renormalized at zero momentum

$$\frac{3}{Ng} = \Delta^{-1}(0, g_0) = \frac{\Gamma(2-d/2)}{2(4\pi)^{d/2}} + \frac{3m^{4-d}}{Ng_0} \equiv \frac{\Gamma(2-d/2)}{2(4\pi)^{d/2}} \frac{N+8}{Ng}, \quad (A3)$$

where we have rescaled the coupling for convenience, generalizing a rather standard three-dimensional prescription. The integration (A2) can be explicitly performed, and one obtains

$$\Delta_r^{-1}(y) = \frac{1}{2} \frac{\Gamma(2-d/2)}{(4\pi)^{d/2}} \left[\left(1 + \frac{y}{4}\right)^{d/2-2} F\left(2 - \frac{d}{2}, \frac{1}{2}, \frac{3}{2}, \frac{y}{y+4}\right) - 1 \right] \equiv \frac{\Gamma(2-d/2)}{2(4\pi)^{d/2}} \delta_r(y), \quad (\text{A4})$$

which is a regular function of d for all $d \leq 4$.

The renormalized $O(1/N)$ contribution to the self-energy [see Eq. (56)] can now be evaluated by the formal expression

$$\phi_1(y, g) = \sigma(y, g) - \sigma(0, g) - y \frac{\partial}{\partial y} \sigma(y, g) \Big|_{y=0}, \quad (\text{A5})$$

$$\sigma(y, g) = m^{2-d} \frac{2(4\pi)^{d/2}}{\Gamma(2-d/2)} \int \frac{d^d p}{(2\pi)^d} \frac{\bar{g}}{1 + \bar{g} \delta_r(p^2/m^2)} \frac{1}{(p+k)^2 + m^2}, \quad (\text{A6})$$

and the subtractions that are symbolically indicated in Eq. (A5) must be done before performing the integration in Eq. (A6) in order to obtain finite quantities in all steps of the derivation. To this purpose, it is convenient to perform first the angular integration, by noticing that

$$\frac{2(4\pi)^{d/2}}{\Gamma(2-d/2)} \int \frac{d^d p}{(2\pi)^d} \frac{m^{2-d}}{(p+k)^2 + m^2} f(p^2/m^2) = 2B(d/2, 2-d/2)^{-1} \int_0^\infty (z)^{d/2-1} dz f(z) h(z, y) \quad (\text{A7})$$

where

$$h(z, y) = \frac{2}{B((d-1)/2, \frac{1}{2})} \int_0^\pi d\theta \frac{(\sin \theta)^{d-2}}{z + y + 1 + 2\sqrt{zy} \cos \theta}. \quad (\text{A8})$$

The subtraction indicated in Eq. (A5) now simply amounts to replacing in Eq. (A6)

$$h(z, y) \rightarrow h(z, y) - h(z, 0) - y \frac{\partial}{\partial y} h(z, y) \Big|_{y=0} = h(z, y) - \frac{1}{1+z} + \frac{y}{(1+z)^2} - \frac{4zy}{d(1+z)^3}. \quad (\text{A9})$$

By replacing \bar{g} with its large- N fixed point value $\bar{g}^* = 1$ in Eq. (A3), one finds the $O(1/N)$ contribution to the scaling function $\hat{g}_0(y)$, which in turn is simply the continuum N -vector model expression of the self-energy. This is the way Eq. (57) is generated, by setting $d=3$ in the general expression.

Equation (A6) is also the starting point for the g and ϵ expansion up to three loops. It is indeed straightforward to obtain a representation of the leading $O(1/N)$ contributions to the self-energy as a power series in g :

$$\phi_1(y, \bar{g}) = -\bar{g}^2 \tilde{\varphi}_2(y) + \bar{g}^3 \tilde{\varphi}_3(y) + O(\bar{g}^4), \quad (\text{A10})$$

where we have defined the functions

$$\tilde{\varphi}_n(y) = (-1)^n \frac{2}{B(d/2, 2-d/2)} \int_0^\infty z^{d/2-1} dz [\delta_r(z)]^{n-1} \left[h(z, y) - \frac{1}{1+z} + \frac{y}{(1+z)^2} - \frac{4zy}{d(1+z)^3} \right], \quad (\text{A11})$$

and we exploited the trivial consequence of the definition Eq. (A11): $\tilde{\varphi}_1(y) \equiv 0$. Restoring the correct dependence on N for arbitrary (and not only very large) values of N in front of the functions φ_2 and φ_3 is now simply a combinatorial problem, whose solution leads to the complete three-loops result for $f(\bar{g}, y) \equiv M_G^{-2} \tilde{G}^{-1}(y) / \tilde{G}^{-1}(0)$:

$$f(\bar{g}, y) = 1 + y + \bar{g}^2 \frac{N+2}{(N+8)^2} \tilde{\varphi}_2(y) + \bar{g}^3 \frac{(N+2)}{(N+8)^2} \tilde{\varphi}_3(y)$$

$$+ O(\bar{g}^4). \quad (\text{A12})$$

We must keep in mind that the functions $\tilde{\varphi}_n(y)$ carry a dependence on the dimensionality d , and the scaling function $\hat{g}_0(y)$ is the value taken by $f(\bar{g}, y)$ when evaluated at the fixed point value \bar{g}^* of the renormalized coupling, where \bar{g}^* is in turn a function of the dimensionality and it is obtained by evaluating the zero of the β function. We may now choose two different strategies. The first simply amounts to fixing d to the physical value we are interested in and replac-

ing \bar{g}^* with the numerical value (possibly evaluated by a higher-order expansion of the β function at fixed dimension). We may, however, decide to expand the functions $\tilde{\varphi}_n(y)$ in the parameter $\epsilon \equiv 4 - d$ around their value at $d=4$, perform a similar expansion for the β function, and then find \bar{g}^* as a series in ϵ [67]:

$$\bar{g}^* = 1 + \frac{3(3N+14)}{(N+8)^2} \epsilon + O(\epsilon^2). \quad (\text{A13})$$

The functions $\varphi_2(y)$ and $\varphi_3(y)$ we have introduced in Sec. III C are strictly related to $\tilde{\varphi}_2(y)$ and $\tilde{\varphi}_3(y)$ calculated for $d=3$, indeed

$$\varphi_2(y) = \tilde{\varphi}_2(y)|_{d=3}, \quad (\text{A14})$$

$$\varphi_3(y) = [\tilde{\varphi}_3(y) - 2\tilde{\varphi}_2(y)]_{d=3}.$$

We now present some details of the calculation to order $1/N$ of the scaling function $\hat{g}_4(y)$. The starting point is

$$g_4(y, M_G) = \frac{1}{\mathcal{N}_4} \int d^d \Omega(\hat{k}) \frac{Q_4(k)}{(k^2)^4} \bar{G}^{-1}(k, M_G), \quad (\text{A15})$$

where $d^d \Omega(\hat{k})$ indicates the normalized measure on the $(d-1)$ -dimensional sphere and

$$\int d^d \Omega(\hat{k}) Q_4(k)^2 = \frac{24(d-1)}{(d+2)^2(d+4)(d+6)} (k^2)^4 \equiv \mathcal{N}_4 (k^2)^4. \quad (\text{A16})$$

Using Eq. (92) we get

$$g_4(y, M_G) = e_{4,0} + \frac{1}{N} \frac{1}{\mathcal{N}_4} \int d^d \Omega(\hat{k}) \frac{Q_4(k)}{(k^2)^4} \times \int \frac{d^d q}{(2\pi)^d} \frac{\bar{\Delta}(q)}{(q+k)^2 + M_G^2}. \quad (\text{A17})$$

From Eq. (21), we get finally

$$\hat{g}_4(y) = 1 - \frac{1}{N} \frac{1}{y^4} \int \frac{d^d q}{(2\pi)^d} [\Delta(q)^2 f_4(q^2, 1) I_1(y, q^2, 1) + \Delta(q) I_2(y, q^2, 1) - (\text{subtr})], \quad (\text{A18})$$

where ‘‘subtr’’ indicates the integrand computed for $y \rightarrow 0$, $\Delta(q)$ is the continuum counterpart of $\bar{\Delta}(q)$:

$$\Delta^{-1}(q) = \frac{1}{2} \frac{\Gamma(2-d/2)}{(4\pi)^{d/2}} \left(\frac{q^2}{4} + M_G^2 \right)^{(d/2)-2} \times F \left[2 - \frac{d}{2}, \frac{1}{2}, \frac{3}{2}, \left(1 + \frac{4M_G^2}{q^2} \right)^{-1} \right], \quad (\text{A19})$$

$$I_1(k^2, q^2, M_G^2) = \frac{1}{\mathcal{N}_4} \int d^d \Omega(\hat{k}) \int d^d \Omega(\hat{q}) \frac{Q_4(k) Q_4(q)}{(q+k)^2 + M_G^2} = (k^2 q^2)^{3/2} F_{d,4}(z), \quad (\text{A20})$$

$$I_2(k^2, q^2, M_G^2) = \frac{1}{\mathcal{N}_4} \int d^d \Omega(\hat{k}) \int d^d \Omega(\hat{q}) \frac{Q_4(k) Q_4(q+k)}{[(q+k)^2 + M_G^2]^2} \quad (\text{A21})$$

$$= -\frac{(k^2)^3}{2q^2} \left[F'_{d,0}(z) + 4 \left(\frac{q^2}{k^2} \right)^{1/2} F'_{d,1}(z) + \frac{6q^2}{k^2} F'_{d,2}(z) + 4 \left(\frac{q^2}{k^2} \right)^{3/2} F'_{d,3}(z) + \left(\frac{q^2}{k^2} \right)^2 F'_{d,4}(z) \right], \quad (\text{A22})$$

where we have defined

$$z = \frac{q^2 + k^2 + M_G^2}{2\sqrt{q^2 k^2}}, \quad (\text{A23})$$

$$F_{d,l}(z) = \frac{2^{(1-d)/2} l! (d-2)!}{\Gamma[(d-1)/2] (d+l-3)!} \times (-1)^l e^{-(d-3)\pi i/2} (z^2 - 1)^{(d-3)/4} Q_{l+(d-3)/2}^{(d-3)/2}(z). \quad (\text{A24})$$

Here $Q_\mu^\nu(z)$ is the associated Legendre function of the second kind (see Ref. [68], Secs. 8.7 and 8.8). Notice that for $l=0$ $F_{d,0}(z) = \sqrt{q^2 k^2} h(q^2/M_G^2, k^2/M_G^2)$. As expected the final result is universal.

APPENDIX B: STRONG-COUPLING EXPANSION OF $G(x)$ ON THE CUBIC LATTICE

Presenting l th-order strong-coupling results for the two-point Green's function would naively imply writing down as many coefficients as the number of lattice sites that can be reached by an l -step random walk starting from the origin (up to discrete lattice symmetries). It is interesting to notice the relationship existing between the number n_l of lattice points (not related by a lattice symmetry) that lie at a given lattice distance l from the origin and the number of independent lattice-symmetric functions $Q_{2m}(k)(k^2)^{l-m}$. One can easily get convinced that, on a hypercubic lattice, the number of functions $Q_{2m}^{(p)}(k)(k^2)^{l-m}$ is the same as the number of monomials of total degree l in the d variables k_i^2 that are not related by a lattice symmetry (that is, the number of independent, homogeneous lattice-symmetric degree- l polynomials in the k_i^2). This number in turn is equal to that of the partitions of l into d ordered non-negative integers, and this is nothing but the number of independent lattice points at a lattice distance l (where ordering ensures independence by elimination of copies obtained by permutation). As a corol-

lary, the relationship $p_l = n_l - n_{l-1}$ holds for arbitrary d on hypercubic lattices [11].

In the case of three-dimensional hypercubic lattices, one can show that $p_l = \lfloor l/6 \rfloor + 1$ with the exception of $l = 6k + 1$ in which case $p_l = k$, while n_l is the integer nearest to $(l + 3)^2/12$ and the sum $\sum_{i=\text{even}}^l n_i$ is the integer nearest to $(l + 4)^3/72$. This would mean roughly $(l + 4)^3/72$ coefficients for the l th-order of the strong-coupling expansion on the cubic lattice. This number can be sensibly reduced (asymptotically by a factor 27 on the cubic lattice), without losing any physical information, by noticing that the inverse two-point function, when represented in coordinate space, involves only points that can be reached by a $\lfloor l/3 \rfloor$ -step random walk. This fact can be traced to the one-particle irreducible nature of the inverse correlation. As a matter of fact, instead of the 93 coefficients needed to represent the 15th-order contributions to $G(x)$, only 8 coefficients are enough for the corresponding contribution to the inverse function $G^{-1}(x)$, which we construct by the following procedure (a similar representation was used for the Ising model in a magnetic field in [6]).

We introduce the equivalence classes of lattice sites under symmetry transformations and choose a representative y for each class: whenever $x \sim y$ then $G(x) = G(y)$. We define the ‘‘form factor’’ of the equivalence class

$$Q(y) = \sum_{x \sim y} e^{ipx}, \quad (\text{B1})$$

and represent the Fourier transform of $G(x)$ according to

$$\tilde{G}(p) = \sum_y Q(y)G(y). \quad (\text{B2})$$

The inverse Fourier transform enjoys the symmetries of $G(x)$ and satisfies the relationships

$$\tilde{G}^{-1}(p) = \sum_x e^{ipx}G^{-1}(x) = \sum_y Q(y)G^{-1}(y). \quad (\text{B3})$$

In practice we exploit the property

$$Q(v)Q(y) = \sum_z n(z;v,y)Q(z), \quad (\text{B4})$$

where

$$n(z;v,y) = \sum_{u \sim v, x \sim y} \delta_{z,u+x} \quad (\text{B5})$$

are integer numbers, and reduce the problem of evaluating $G^{-1}(y)$ to that of solving the linear system of equations

$$\sum_v G^{-1}(v)M(v,z) = \delta_{z,0}, \quad (\text{B6})$$

where

$$M(v,z) = \sum_y G(y)n(z;v,y). \quad (\text{B7})$$

When expanding in powers of β , the system takes a triangular structure and, as expected, it admits a solution whose nontrivial terms are only those corresponding to the equivalence classes of sites that can be reached by $\lfloor l/3 \rfloor$ random steps.

Solutions for $G^{-1}(x)$ can be found for arbitrary N . In Table VII we only exhibit $G^{-1}(x)$ for $N=0, 1, 2, 3, 4$, and 16. We choose a representative of the equivalence class by the prescription $x_1 \geq x_2 \geq x_3 \geq 0$. We may notice as a general feature that in the class represented by $x_1 > 1, x_2 = x_3 = 0$ the first nontrivial contribution is of order $3x_1 + 2$ ($3x_1 + 4$ when $N=1$). When $N=0,1$ a number of seemingly nontrivial coefficients turn out to be zero.

APPENDIX C: STRONG-COUPLING SERIES OF χ AND m_2 ON THE DIAMOND LATTICE

On the diamond lattice we have calculated the strong-coupling expansion of $G(x)$ up to 21st order. In the characterlike approach [42], the possibility of reaching larger orders than on the cubic lattice is related to the smaller coordination number. However, longer series do not necessarily mean that more precise results can be obtained from their analysis. This is essentially related to the approach to the asymptotic regime of the corresponding strong-coupling expansion, which is expected to occur later on lattices with smaller coordination number. 21st-order series on the diamond lattice provide estimates of the exponents γ and ν , which are, as we shall see for $N=1,2,3$, substantially consistent with the results obtained by analyzing series on other lattices [see, for example, Ref. [43] where series to $O(\beta^{21})$ for the cubic and bcc lattice have been presented and analyzed], but less precise.

On the diamond lattice we have defined a mass-gap estimator according to the following procedure. Let us parameterize the Cartesian coordinates of the sites \vec{x} of the diamond lattice in the form $\vec{x} = \sum l_i \vec{\eta}_i + p \vec{\eta}_p$, $l_i \in \mathbb{Z}$, $p=0,1$, $\vec{\eta}_p = (1/\sqrt{3})(1,1,1)$, $\vec{\eta}_1 = (2/\sqrt{3})(0,1,1)$, $\vec{\eta}_2 = (2/\sqrt{3})(1,0,1)$, $\vec{\eta}_3 = (2/\sqrt{3})(1,1,0)$. One may then consider the wall-wall correlation function defined constructing walls orthogonal to $\vec{w} = (1/\sqrt{2})(-1,1,0)$, which is the direction orthogonal to two among the links starting from a site. We define

$$G_w(t \equiv \vec{x} \cdot \vec{w}) = \sum_{t=\text{cst}} G(\vec{x}), \quad (\text{C1})$$

where the sum is performed over all sites with the same $t \equiv \vec{x} \cdot \vec{w} = (2/\sqrt{3})(l_1 - l_2)$. One can prove that $G_w(t)$ enjoys the property of exponentiation. The mass gap μ can be extracted from the long-distance behavior of $G_w(t)$. For $t \gg 1$ $G_w(t) \propto e^{-\mu t}$. In view of a strong-coupling analysis, it is convenient to use the quantity

TABLE VII. Coefficients of the strong-coupling expansion of $G^{-1}(x)$ on the cubic lattice. The representative of each equivalence class is chosen by $x_1 \geq x_2 \geq x_3 \geq 0$. l indicates the order.

x_1	x_2	x_3	l	$N=0$	$N=1$	$N=2$	$N=3$	$N=4$	$N=16$
0	0	0	0	1	1	1	1	1	1
1	0	0	1	-1	-1	-1	-1	-1	-1
0	0	0	2	6	6	6	6	6	6
1	0	0	3	-1	$-\frac{2}{3}$	$-\frac{1}{2}$	$-\frac{2}{5}$	$-\frac{1}{3}$	$-\frac{1}{9}$
0	0	0	4	30	26	24	$\frac{114}{5}$	22	$\frac{58}{3}$
1	0	0	5	-13	$-\frac{122}{15}$	$-\frac{35}{6}$	$-\frac{158}{35}$	$-\frac{11}{3}$	$-\frac{49}{45}$
0	0	0	6	366	$\frac{4204}{15}$	$\frac{479}{2}$	$\frac{37788}{175}$	$\frac{602}{3}$	$\frac{20674}{135}$
1	1	0	6	2	0	$-\frac{1}{2}$	$-\frac{16}{25}$	$-\frac{2}{3}$	$-\frac{10}{27}$
1	0	0	7	-197	$-\frac{33604}{315}$	$-\frac{1123}{16}$	$-\frac{44812}{875}$	$-\frac{1793}{45}$	$-\frac{132961}{13365}$
0	0	0	8	5022	$\frac{348266}{105}$	$\frac{10265}{4}$	$\frac{1882494}{875}$	$\frac{9454}{5}$	$\frac{5012914}{4455}$
1	1	0	8	24	-16	$-\frac{65}{3}$	$-\frac{18576}{875}$	$-\frac{176}{9}$	$-\frac{1912}{243}$
2	0	0	8	4	0	-1	$-\frac{32}{25}$	$-\frac{4}{3}$	$-\frac{20}{27}$
1	0	0	9	-2889	$-\frac{3805202}{2835}$	$-\frac{191503}{240}$	$-\frac{5197194}{9625}$	$-\frac{53443}{135}$	$-\frac{2940271}{40095}$
1	1	1	9	6	-8	$-\frac{57}{4}$	$-\frac{408}{25}$	$-\frac{50}{3}$	$-\frac{778}{81}$
2	1	0	9	-1	0	$-\frac{1}{4}$	$-\frac{56}{125}$	$-\frac{5}{9}$	$-\frac{115}{243}$
0	0	0	10	76062	$\frac{211434604}{4725}$	$\frac{3833513}{120}$	$\frac{8484905796}{336875}$	21078	$\frac{1969903274}{200475}$
1	1	0	10	258	$-\frac{952}{3}$	$-\frac{24851}{72}$	$-\frac{1878984}{6125}$	$-\frac{35786}{135}$	$-\frac{10005158}{120285}$
2	0	0	10	116	-24	$-\frac{583}{12}$	$-\frac{45272}{875}$	$-\frac{1340}{27}$	$-\frac{49060}{2187}$
1	0	0	11	-45357	$-\frac{2874597004}{155925}$	$-\frac{42875903}{4320}$	$-\frac{949030894596}{153278125}$	$-\frac{12000923}{2835}$	$-\frac{34414485049}{70366725}$
1	1	1	11	72	-240	$-\frac{2635}{8}$	$-\frac{7370064}{21875}$	$-\frac{8636}{27}$	$-\frac{915928}{6561}$
2	1	0	11	-15	-24	$-\frac{133}{3}$	$-\frac{1142304}{21875}$	$-\frac{4369}{81}$	$-\frac{605717}{19683}$
3	0	0	11	0	0	$-\frac{1}{2}$	$-\frac{96}{125}$	$-\frac{8}{9}$	$-\frac{160}{243}$
0	0	0	12	1230462	$\frac{101355262012}{155925}$	$\frac{77441167}{180}$	$\frac{49050932335452}{153278125}$	$\frac{26831414}{105}$	$\frac{2184687177202}{23455575}$
1	1	0	12	2460	$-\frac{254288}{45}$	$-\frac{470591}{90}$	$-\frac{49887963408}{11790625}$	$-\frac{1379716}{405}$	$-\frac{3960090404}{5412825}$
2	0	0	12	1944	-896	$-\frac{43417}{36}$	$-\frac{35397888}{30625}$	$-\frac{423976}{405}$	$-\frac{427970056}{1082565}$
2	1	1	12	0	-24	$-\frac{393}{8}$	$-\frac{39096}{625}$	$-\frac{616}{9}$	$-\frac{35456}{729}$
2	2	0	12	0	0	$-\frac{7}{8}$	$-\frac{1104}{625}$	$-\frac{64}{27}$	$-\frac{5600}{2187}$
3	1	0	12	0	0	$-\frac{1}{8}$	$-\frac{192}{625}$	$-\frac{4}{9}$	$-\frac{400}{729}$
1	0	0	13	-745189	$-\frac{1629512844964}{6081075}$	$-\frac{2652479497}{20160}$	$-\frac{11642408503972}{153278125}$	$-\frac{45795581}{945}$	$-\frac{470143113389}{164189025}$
1	1	1	13	-678	$-\frac{28728}{5}$	$-\frac{614743}{96}$	$-\frac{903833736}{153125}$	$-\frac{704326}{135}$	$-\frac{600662842}{360855}$
2	1	0	13	-476	-1120	$-\frac{437317}{288}$	$-\frac{241988968}{153125}$	$-\frac{616348}{405}$	$-\frac{765424036}{1082565}$
3	0	0	13	0	-24	$-\frac{1337}{24}$	$-\frac{1500984}{21875}$	$-\frac{5864}{81}$	$-\frac{868672}{19683}$
0	0	0	14	20787102	$\frac{140597688722408}{14189175}$	$\frac{245838046393}{40320}$	$\frac{298587843693288}{69671875}$	$\frac{46330187578}{14175}$	$\frac{15116615635191578}{16254713475}$
1	1	0	14	17378	$-\frac{18013984}{189}$	$-\frac{1330017637}{17280}$	$-\frac{43519901193056}{766390625}$	$-\frac{1801726502}{42525}$	$-\frac{35538833424418}{6966305775}$
2	0	0	14	29088	-20744	$-\frac{8258723}{360}$	$-\frac{1192015283352}{58953125}$	$-\frac{21049024}{1215}$	$-\frac{253843671008}{48715425}$
2	1	1	14	0	-1456	$-\frac{54377}{24}$	$-\frac{278755152}{109375}$	$-\frac{628336}{243}$	$-\frac{253847360}{177147}$

TABLE VII. (Continued).

x_1	x_2	x_3	l	$N=0$	$N=1$	$N=2$	$N=3$	$N=4$	$N=16$
2	2	0	14	32	-144	$-\frac{2425}{8}$	$-\frac{42568048}{109375}$	$-\frac{103640}{243}$	$-\frac{52536352}{177147}$
3	1	0	14	-2	-32	$-\frac{206}{3}$	$-\frac{1427216}{15625}$	$-\frac{8294}{81}$	$-\frac{4477778}{59049}$
4	0	0	14	0	0	$-\frac{1}{4}$	$-\frac{288}{625}$	$-\frac{16}{27}$	$-\frac{1280}{2187}$
1	0	0	15	-12672757	$-\frac{2597638257068408}{638512875}$	$-\frac{3549389785799}{1935360}$	$-\frac{64161784918165784}{65143203125}$	$-\frac{25033873577}{42525}$	$-\frac{305112195055173211}{24138249510375}$
1	1	1	15	-48624	$-\frac{7317712}{63}$	$-\frac{317601607}{2880}$	$-\frac{555516678768}{6015625}$	$-\frac{92268604}{1215}$	$-\frac{2363546376656}{146146275}$
2	1	0	15	-16428	$-\frac{154436}{5}$	$-\frac{605816971}{17280}$	$-\frac{1403163936156}{42109375}$	$-\frac{110231638}{3645}$	$-\frac{4917927730652}{438438825}$
2	2	1	15	30	-232	$-\frac{8199}{16}$	$-\frac{2141976}{3125}$	$-\frac{20954}{27}$	$-\frac{4066162}{6561}$
3	0	0	15	-400	-1752	$-\frac{192203}{72}$	$-\frac{11182524776}{3828125}$	$-\frac{10592632}{3645}$	$-\frac{133926221584}{87687765}$
3	1	1	15	-10	-48	$-\frac{1681}{16}$	$-\frac{90832}{625}$	$-\frac{506}{3}$	$-\frac{953338}{6561}$
3	2	0	15	-1	0	$-\frac{13}{8}$	$-\frac{11936}{3125}$	$-\frac{151}{27}$	$-\frac{50545}{6561}$
4	1	0	15	0	0	$-\frac{1}{16}$	$-\frac{648}{3125}$	$-\frac{28}{81}$	$-\frac{11840}{19683}$

$$M_d^2 \equiv \frac{4}{3} (\cosh \sqrt{\frac{3}{2}} \mu - 1), \quad (C2)$$

which has the property $M_d \rightarrow \mu$ for $\mu \rightarrow 0$ and has a regular strong-coupling series. In the large- N limit and for $\beta \leq \beta_c$ $M_d^2/M_G^2 = 1$.

In the following we report the 21st-order strong-coupling series of χ and m_2 calculated on the diamond lattice, for $N = 1, 2, 3$. 27th-order strong-coupling series for $N=0$, i.e., for the self-avoiding walk model, can be found in Ref. [45].

1. $N=1$

$$\begin{aligned} \chi = & 1 + 4\beta + 12\beta^2 + \frac{104\beta^3}{3} + 100\beta^4 + \frac{4328\beta^5}{15} + \frac{12128\beta^6}{15} + \frac{711328\beta^7}{315} + \frac{132452\beta^8}{21} + \frac{49894088\beta^9}{2835} \\ & + \frac{230044448\beta^{10}}{4725} + \frac{20986492048\beta^{11}}{155925} + \frac{11593048528\beta^{12}}{31185} + \frac{6239638466896\beta^{13}}{6081075} + \frac{40044715794736\beta^{14}}{14189175} \\ & + \frac{381115667726672\beta^{15}}{49116375} + \frac{907261838473556\beta^{16}}{42567525} + \frac{635228192216156408\beta^{17}}{10854718875} + \frac{5223277546855685888\beta^{18}}{32564156625} \\ & + \frac{815815904018756584288\beta^{19}}{1856156927625} + \frac{744572898253973823856\beta^{20}}{618718975875} + \frac{642020997051581736673936\beta^{21}}{194896477400625} + O(\beta^{22}). \end{aligned} \quad (C3)$$

$$\begin{aligned} m_2 = & 4\beta + 32\beta^2 + \frac{488\beta^3}{3} + \frac{2048\beta^4}{3} + \frac{38888\beta^5}{15} + \frac{417664\beta^6}{45} + \frac{10027936\beta^7}{315} + \frac{33306368\beta^8}{315} + \frac{971601608\beta^9}{2835} \\ & + \frac{15453950464\beta^{10}}{14175} + \frac{532482065296\beta^{11}}{155925} + \frac{4939730085376\beta^{12}}{467775} + \frac{196443743845456\beta^{13}}{6081075} + \frac{4168605624019328\beta^{14}}{42567525} \\ & + \frac{188065240470724112\beta^{15}}{638512875} + \frac{561744708980235008\beta^{16}}{638512875} + \frac{28352355075085440248\beta^{17}}{10854718875} \\ & + \frac{57966531061027107328\beta^{18}}{7514805375} + \frac{42081281751167641189216\beta^{19}}{1856156927625} + \frac{615759333006052918694656\beta^{20}}{9280784638125} \\ & + \frac{37696556941296724618984336\beta^{21}}{194896477400625} + O(\beta^{22}). \end{aligned} \quad (C4)$$

We have analyzed the series of χ by using the $[m/l/k]$ first-order IA's with

$$m+l+k+2=21, \quad (C5)$$

$$[(n-2)/3]-2 \leq m, l, k \leq [(n-2)/3]+2.$$

We have obtained $\beta_c=0.3697(1)$ and $\gamma=1.238(14)$. An estimate of γ can be also obtained by applying the CPRM to the series χ^2 and χ , as explained in Sec. IV F. By employing biased IA's, one finds $\gamma=1.253(4)$. By applying the CPRM to the series m_2 and χ , and using biased IA's, one finds $\nu=0.645(4)$. These values of γ and ν are slightly larger than the available estimates obtained by other techniques (field-theoretical approaches give $\gamma \approx 1.240$ and $\nu \approx 0.630$), or strong-coupling expansion on other lattices, but not totally inconsistent. One should not forget that the reported error does not take into account the systematic errors due to confluent singularities, but is just the spread of the results of the various IA's indicated in Eq. (C5).

2. $N=2$

$$\begin{aligned} \chi = & 1 + 4\beta + 12\beta^2 + 34\beta^3 + 96\beta^4 + \frac{814\beta^5}{3} + 743\beta^6 + \frac{24145\beta^7}{12} + \frac{10925\beta^8}{2} + \frac{889703\beta^9}{60} + \frac{2387483\beta^{10}}{60} \\ & + \frac{22968773\beta^{11}}{216} + \frac{25617551\beta^{12}}{90} + \frac{11516036093\beta^{13}}{15120} + \frac{40849680041\beta^{14}}{20160} + \frac{520550507027\beta^{15}}{96768} + \frac{3457894675397\beta^{16}}{241920} \\ & + \frac{495995794312009\beta^{17}}{13063680} + \frac{2188572410969059\beta^{18}}{21772800} + \frac{173608313274399461\beta^{19}}{653184000} + \frac{76543471229019871\beta^{20}}{108864000} \\ & + \frac{5344313242348050991\beta^{21}}{2874009600} + O(\beta^{22}). \end{aligned} \quad (C6)$$

$$\begin{aligned} m_2 = & 4\beta + 32\beta^2 + 162\beta^3 + 672\beta^4 + \frac{7534\beta^5}{3} + \frac{26488\beta^6}{3} + \frac{356305\beta^7}{12} + \frac{289444\beta^8}{3} + \frac{18326503\beta^9}{60} + \frac{42659326\beta^{10}}{45} \\ & + \frac{3125910649\beta^{11}}{1080} + \frac{1176454982\beta^{12}}{135} + \frac{78423473449\beta^{13}}{3024} + \frac{577822206313\beta^{14}}{7560} + \frac{108069034519903\beta^{15}}{483840} \\ & + \frac{58770348791597\beta^{16}}{90720} + \frac{24384512261505001\beta^{17}}{13063680} + \frac{43660988509648999\beta^{18}}{8164800} + \frac{9954936929950180901\beta^{19}}{653184000} \\ & + \frac{1764942584095467281\beta^{20}}{40824000} + \frac{1754883256361403082267\beta^{21}}{14370048000} + O(\beta^{22}). \end{aligned} \quad (C7)$$

By performing an IA analysis of the series of χ , one finds $\beta_c=0.3845(2)$ and $\gamma=1.33(2)$. By applying the CPRM to the series χ^2 and χ , and employing biased IA's, one finds $\gamma=1.34(1)$. By applying the CPRM to the series m_2 and χ , and using biased IA's, one finds $\nu=0.689(8)$. These results are substantially consistent with the available estimates of γ obtained on other lattices and by other approaches (see, e.g., Refs. [43] and [2]).

3. $N=3$

$$\begin{aligned} \chi = & 1 + 4\beta + 12\beta^2 + \frac{168\beta^3}{5} + \frac{468\beta^4}{5} + \frac{9144\beta^5}{35} + \frac{123456\beta^6}{175} + \frac{65568\beta^7}{35} + \frac{873708\beta^8}{175} + \frac{128270568\beta^9}{9625} \\ & + \frac{11818853472\beta^{10}}{336875} + \frac{2007117038928\beta^{11}}{21896875} + \frac{5262987995856\beta^{12}}{21896875} + \frac{1973906542032\beta^{13}}{3128125} + \frac{25696714370736\beta^{14}}{15640625} \\ & + \frac{277395071474138256\beta^{15}}{65143203125} + \frac{5048136975344060076\beta^{16}}{456002421875} + \frac{1747312876419771883176\beta^{17}}{60648322109375} \\ & + \frac{35523883350405253078656\beta^{18}}{476522530859375} + \frac{3207088211587054727672352\beta^{19}}{16678288580078125} + \frac{8294186293466843988864336\beta^{20}}{16678288580078125} \\ & + \frac{285612671193686662161552\beta^{21}}{221862716796875} + O(\beta^{22}). \end{aligned} \quad (C8)$$

$$\begin{aligned}
m_2 = & 4\beta + 32\beta^2 + \frac{808\beta^3}{5} + \frac{3328\beta^4}{5} + \frac{17240\beta^5}{7} + \frac{1498496\beta^6}{175} + \frac{4978592\beta^7}{175} + \frac{15959296\beta^8}{175} + \frac{391158744\beta^9}{1375} \\
& + \frac{292871549952\beta^{10}}{336875} + \frac{8170771755824\beta^{11}}{3128125} + \frac{169326765636096\beta^{12}}{21896875} + \frac{495146153921968\beta^{13}}{21896875} \\
& + \frac{7166586778308992\beta^{14}}{109484375} + \frac{1747788082514945008\beta^{15}}{9306171875} + \frac{243755148694999429888\beta^{16}}{456002421875} \\
& + \frac{91627122308762345759912\beta^{17}}{60648322109375} + \frac{2022510678813614989101568\beta^{18}}{476522530859375} + \frac{197787508407138584345236512\beta^{19}}{16678288580078125} \\
& + \frac{21994072978677629556242688\beta^{20}}{667131543203125} + \frac{34998691725014346631615751056\beta^{21}}{383600637341796875} + O(\beta^{22}). \tag{C9}
\end{aligned}$$

By performing an IA analysis of the series of χ , one finds $\beta_c = 0.3951(2)$ and $\gamma = 1.42(2)$. We mention that singularities approximately as far to the origin as β_c have been detected by our analysis, indeed we found two singularities at $\beta \approx \pm i0.39$. By applying the CPRM to the series χ^2 and χ , and employing biased IA's, one obtains $\gamma = 1.42(1)$. By applying the CPRM to the series m_2 and χ , and using biased IA's, one finds $\nu = 0.726(4)$. These results are slightly larger (and less precise) than the values obtained on other lattices (see, e.g., Ref. [43]), or by other techniques (see, e.g., Ref. [2]), but substantially consistent.

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